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On the Attached Primes and Shifted Localization Principle for Local Cohomology Modules*

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Abstract. Let $(R, \mathfrak{m})$ be a Noetherian local ring and M a finitely generated R-module. For an integer $i \geq 0$, the Artinian i-th local cohomology module $H^i_{\mathfrak{m}}(M)$ is said to satisfy the shifted localization principle if

$$\operatorname{Att}_{R_\mathfrak{p}}(H^{i-\dim R/\mathfrak{p}}_{\mathfrak{p}R_\mathfrak{p}}(M_\mathfrak{p})) = \{\mathfrak{q}R_\mathfrak{p} \mid \mathfrak{q} \in \operatorname{Att}_R(H^i_{\mathfrak{m}}(M)),\ \mathfrak{q} \subseteq \mathfrak{p}\}$$

for all $\mathfrak{p} \in \operatorname{Spec}(R)$. In this paper we study the attached primes of $H^i_{\mathfrak{m}}(M)$ and give some conditions for $H^i_{\mathfrak{m}}(M)$ to satisfy the shifted localization principle.

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1 Introduction

Throughout this paper, let $(R, \mathfrak{m})$ be a Noetherian local ring and let M be a finitely generated R-module with $\dim M = d$. It is clear that

$$\operatorname{Ass}_{R_\mathfrak{p}}(M_\mathfrak{p}) = \{\mathfrak{q}R_\mathfrak{p} \mid \mathfrak{q} \in \operatorname{Ass}_R(M),\ \mathfrak{q} \subseteq \mathfrak{p}\}$$

for all $\mathfrak{p} \in \operatorname{Spec}(R)$. We consider the analogous formula for attached primes of the Artinian i-th local cohomology module $H^i_{\mathfrak{m}}(M)$ as follows:

$$\operatorname{Att}_{R_\mathfrak{p}}(H^{i-\dim R/\mathfrak{p}}_{\mathfrak{p}R_\mathfrak{p}}(M_\mathfrak{p})) = \{\mathfrak{q}R_\mathfrak{p} \mid \mathfrak{q} \in \operatorname{Att}_R(H^i_{\mathfrak{m}}(M)),\ \mathfrak{q} \subseteq \mathfrak{p}\}$$

for all $\mathfrak{p} \in \operatorname{Spec}(R)$. We say that $H^i_{\mathfrak{m}}(M)$ satisfies the *shifted localization principle* if this formula holds true. In general, $H^i_{\mathfrak{m}}(M)$ satisfies the *weak general shifted localization principle*, i.e.,

$$\operatorname{Att}_{R_\mathfrak{p}}(H^{i-\dim R/\mathfrak{p}}_{\mathfrak{p}R_\mathfrak{p}}(M_\mathfrak{p})) \subseteq \{\mathfrak{q}R_\mathfrak{p} \mid \mathfrak{q} \in \operatorname{Att}_R(H^i_{\mathfrak{m}}(M)),\ \mathfrak{q} \subseteq \mathfrak{p}\}$$

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for all $\mathfrak{p} \in \mathrm{Spec}(R)$ (cf. [11, Theorem 4.8], see also [2, 11.3.8]). In case R is a quotient of a Gorenstein local ring, $H^i_{\mathfrak{m}}(M)$ always satisfies the shifted localization principle (cf. [11, Theorem 3.7], see also [2, 11.3.2]). However, the shifted localization principle is not valid in general. For example, let $(R, \mathfrak{m})$ be the Noetherian local domain of dimension 2 constructed by Ferrand and Raynaud [7] such that $\widehat{R}$ has an associated prime ideal $\widehat{\mathfrak{p}}$ of dimension 1. Then $H^1_{\mathfrak{m}}(R)$ does not satisfy the shifted localization principle (cf. [2, 11.3.14]). Moreover, if $(R, \mathfrak{m})$ is a Noetherian local domain of dimension 1 that is not a homomorphic image of a Gorenstein local ring (such a ring exists by [7]), then it is clear that $H^i_{\mathfrak{m}}(M)$ satisfies the shifted localization principle for any finitely generated R-module M and any integer i. Therefore, it is natural to ask under which conditions the shifted localization principle is valid for $H^i_{\mathfrak{m}}(M)$.

The purpose of this paper is to study the attached primes of $H^i_{\mathfrak{m}}(M)$ in order to give some conditions for $H^i_{\mathfrak{m}}(M)$ to satisfy the shifted localization principle.

For each ideal I of R, we denote by $\mathrm{Var}(I)$ the set of all prime ideals of R containing I. Before stating the main results, we recall the following property on an Artinian R-module A, which was considered first by Cuong and Nhan [5]:

$$\mathrm{Ann}_R(0 :_A \mathfrak{p}) = \mathfrak{p} \text{ for all } \mathfrak{p} \in \mathrm{Var}(\mathrm{Ann}_R A). \qquad (*)$$

If R is complete with respect to the $\mathfrak{m}$-adic topology, it follows by the Matlis duality that the property $(*)$ is satisfied for all Artinian R-modules A. If R is universally catenary and all its formal fibres are Cohen-Macaulay, then $H^i_{\mathfrak{m}}(M)$ satisfies the property $(*)$ for any integer i (cf. [9, Corollary 3.2]). However, there exists a local cohomology module $H^1_{\mathfrak{m}}(R)$ that does not satisfy the property $(*)$ (cf. [5, Example 4.3]). It should be mentioned that if R is not complete, then the study of the property $(*)$ for $H^i_{\mathfrak{m}}(M)$ is important since it gives a lot of information on the module M and the base ring R (cf. [4, 5, 9, 10, 12]). Also, the results in this paper show that the property $(*)$ is closely related to the shifted localization principle.

Note that $H^d_{\mathfrak{m}}(M)$ satisfies the property $(*)$ if and only if the ring $R/\mathrm{Ann}_R H^d_{\mathfrak{m}}(M)$ is catenary (cf. [4]). Together with this fact, our first main result gives some characterizations for the top local cohomology to satisfy the shifted localization principle.

Theorem 1.1. *The following statements are equivalent:*

(i) $H^d_{\mathfrak{m}}(M)$ *satisfies the shifted localization principle.*
(ii) *The ring* $R/\mathrm{Ann}_R H^d_{\mathfrak{m}}(M)$ *is catenary.*
(iii) $H^d_{\mathfrak{m}}(M)$ *satisfies the property* $(*)$.
(iv) $H^{d-\dim R/\mathfrak{p}}_{\mathfrak{p}R_{\mathfrak{p}}}(M_{\mathfrak{p}})$ *satisfies the shifted localization principle for all* $\mathfrak{p} \in \mathrm{Supp}(M)$.
(v) $H^{d-\dim R/\mathfrak{p}}_{\mathfrak{p}R_{\mathfrak{p}}}(M_{\mathfrak{p}})$ *satisfies the property* $(*)$ *for all* $\mathfrak{p} \in \mathrm{Supp}(M)$.

For the lower levels $i < d$, we show how the property $(*)$ is related to the shifted localization principle for minimal attached primes and how it behaves under localization.

Consider the following conditions:

(a) $\min \operatorname{Att}_{R_\mathfrak{p}}(H^{i-\dim R/\mathfrak{p}}_{\mathfrak{p}R_\mathfrak{p}}(M_\mathfrak{p})) = \{\mathfrak{q}R_\mathfrak{p} \mid \mathfrak{q} \in \min \operatorname{Att}_R(H^i_\mathfrak{m}(M)),\ \mathfrak{q} \subseteq \mathfrak{p}\}$ for all $\mathfrak{p} \in \operatorname{Spec}(R)$.

(b) $H^i_\mathfrak{m}(M)$ satisfies the property $(*)$.

(c) $H^{i-\dim R/\mathfrak{p}}_{\mathfrak{p}R_\mathfrak{p}}(M_\mathfrak{p})$ satisfies the property $(*)$ for all $\mathfrak{p} \in \operatorname{Supp}(M)$.

Theorem 1.2. *Let $i \geq 0$ be an integer.*

(i) *If $R/\operatorname{Ann}_R M$ is catenary, then the conditions* (a), (b), (c) *are equivalent.*

(ii) *If $H^i_\mathfrak{m}(M)$ satisfies the shifted localization principle, then the conditions* (a) *and* (b) *are satisfied.*

Following Brodmann and Sharp [3], the i-th *pseudo support* of M is defined as

$$\operatorname{Psupp}^i_R(M) = \{\mathfrak{p} \in \operatorname{Spec}(R) \mid H^{i-\dim R/\mathfrak{p}}_{\mathfrak{p}R_\mathfrak{p}}(M_\mathfrak{p}) \neq 0\}.$$

If R is universally catenary and all its formal fibres are Cohen-Macaulay, then $\operatorname{Psupp}^i_R(M) = \operatorname{Var}(\operatorname{Ann}_R H^d_\mathfrak{m}(M))$ is a closed subset of $\operatorname{Spec}(R)$ in the Zariski topology, but in general $\operatorname{Psupp}^i_R(M)$ is a proper subset of $\operatorname{Var}(\operatorname{Ann}_R H^d_\mathfrak{m}(M))$ and is not closed (cf. [3] and [9]). Our last main result gives some information on the pseudo supports of M and the attached primes of the local cohomology modules $H^i_\mathfrak{m}(M)$.

Theorem 1.3. *Let $i \geq 0$ be an integer. Then*

(i) $\operatorname{Psupp}^i_R(M) \setminus \bigcup_{j=0}^{i-1} \operatorname{Psupp}^j_R(M) = \{\mathfrak{p} \in \operatorname{Supp}(M) \mid \operatorname{depth} M_\mathfrak{p} + \dim R/\mathfrak{p} = i\}$.

(ii) *If $H^i_\mathfrak{m}(M)$ satisfies the property $(*)$, then*

$$\begin{aligned} &\operatorname{Att}_R(H^i_\mathfrak{m}(M)) \setminus \textstyle\bigcup_{j=0}^{i-1} \operatorname{Psupp}^j_R(M) \\ = \ &\{\mathfrak{p} \in \operatorname{Att}_R(H^i_\mathfrak{m}(M)) \mid \operatorname{depth} M_\mathfrak{p} + \dim R/\mathfrak{p} = i\}. \end{aligned}$$

(iii) *If $H^j_\mathfrak{m}(M)$ satisfies the property $(*)$ for all $j \leq i$, then*

$$\begin{aligned} &\min \operatorname{Att}_R(H^i_\mathfrak{m}(M)) \setminus \textstyle\bigcup_{j=0}^{i-1} \operatorname{Var}(\operatorname{Ann}_R H^j_\mathfrak{m}(M)) \\ = \ &\min\{\mathfrak{p} \in \operatorname{Supp}(M) \mid \operatorname{depth} M_\mathfrak{p} + \dim R/\mathfrak{p} = i\}. \end{aligned}$$

The formula in Theorem 1.3(iii) is known if R is a quotient of a Gorenstein local ring (cf. [2, 11.3.12]). Here we show that this formula is still valid under the weaker assumption that $H^j_\mathfrak{m}(M)$ satisfies the property $(*)$ for all $j \leq i$.

This paper is divided into three sections. In the next section we present some preliminaries on the pseudo supports of M, the attached primes and the property $(*)$ for local cohomology modules. The proofs of the main results are given in the last section.

2 Preliminaries

The theory of secondary representation for Artinian modules was introduced by Macdonald [8]. Let A be an Artinian R-module. Then A has a minimal secondary representation $A = A_1 + \cdots + A_n$, where A_i is $\mathfrak{p}_i$-secondary. The set $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$ is independent of the choice of the minimal secondary representation of A. This set is called the *set of attached prime ideals* of A, and denoted by $\mathrm{Att}_R(A)$. Note that $A \neq 0$ if and only if $\mathrm{Att}_R(A) \neq \emptyset$. Moreover, A has the natural structure as an $\widehat{R}$-module. With this structure, a subset of A is an R-submodule of A if and only if it is an $\widehat{R}$-submodule. Therefore, A is an Artinian $\widehat{R}$-module. The following properties for attached primes of Artinian modules can be found in [8] and [2, 8.2.5].

Lemma 2.1. *Let A be an Artinian R-module. Then we have:*
(i) $\min \mathrm{Att}_R(A) = \min \mathrm{Var}(\mathrm{Ann}_R A)$.
(ii) $\mathrm{Att}_R(A) = \{\, \widehat{\mathfrak{p}} \cap R \mid \widehat{\mathfrak{p}} \in \mathrm{Att}_{\widehat{R}}(A)\}$.

Note that the role of $\mathrm{Psupp}^i_R(M)$ for the Artinian R-module $A = H^i_{\mathfrak{m}}(M)$ is in some sense similar to that of $\mathrm{Supp}(L)$ for a finitely generated R-module L (cf. [3] and [10]). However, although we always have $\mathrm{Supp}(L) = \mathrm{Var}(\mathrm{Ann}_R L)$, the analogous equality $\mathrm{Psupp}^i_R(M) = \mathrm{Var}(\mathrm{Ann}_R H^i_{\mathfrak{m}}(M))$ is not valid in general. The following connection between these sets in the general case is given in [6, 9]:

Lemma 2.2. *Let $i \geq 0$ be an integer. Then* $\mathrm{Psupp}^i_R(M) \subseteq \mathrm{Var}(\mathrm{Ann}_R H^i_{\mathfrak{m}}(M))$.

Lemma 2.3. [4, 9] *The following statements are true:*
(i) *$H^d_{\mathfrak{m}}(M)$ satisfies the property $(*)$ if and only if the ring $R/\mathrm{Ann}_R H^d_{\mathfrak{m}}(M)$ is catenary.*
(ii) *For each integer $i \geq 0$, $H^i_{\mathfrak{m}}(M)$ satisfies the property $(*)$ if and only if $\mathrm{Psupp}^i(M) = \mathrm{Var}(\mathrm{Ann}_R H^i_{\mathfrak{m}}(M))$.*
(iii) *If the ring $R/\mathrm{Ann}_R M$ is universally catenary and all its formal fibres are Cohen-Macaulay, then $H^i_{\mathfrak{m}}(M)$ satisfies the property $(*)$ for all i.*
(iv) *If $H^i_{\mathfrak{m}}(M)$ satisfies the property $(*)$ for all $i < d$, then $R/\mathfrak{p}$ is unmixed (i.e., $\dim \widehat{R}/\widehat{\mathfrak{p}} = \dim R/\mathfrak{p}$ for all $\widehat{\mathfrak{p}} \in \mathrm{Ass}(\widehat{R}/\mathfrak{p}\widehat{R})$) for all $\mathfrak{p} \in \mathrm{Ass}_R(M)$, and the ring $R/\mathrm{Ann}_R M$ is universally catenary.*

3 Main Results

Lemma 3.1. *Let $i \geq 0$ be an integer. If the ring $R/\mathrm{Ann}_R H^i_{\mathfrak{m}}(M)$ is catenary, then*

$$\mathrm{Psupp}^{i-\dim R/\mathfrak{p}}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) \supseteq \{\mathfrak{q}R_{\mathfrak{p}} \mid \mathfrak{q} \in \mathrm{Psupp}^i_R(M),\ \mathfrak{q} \subseteq \mathfrak{p}\}$$

for all $\mathfrak{p} \in \mathrm{Spec}(R)$. The equality holds true if the ring $R/\mathrm{Ann}_R M$ is catenary.

Proof. Assume that $R/\mathrm{Ann}_R H^i_{\mathfrak{m}}(M)$ is catenary. Let $\mathfrak{p} \in \mathrm{Spec}(R)$. If $\mathfrak{q} \subseteq \mathfrak{p}$ and $\mathfrak{q} \in \mathrm{Psupp}^i_R(M)$, then by Lemma 2.2, we have $\mathfrak{q} \in \mathrm{Var}(\mathrm{Ann}_R H^i_{\mathfrak{m}}(M))$. So

$$\mathfrak{p} \supseteq \mathfrak{q} \supseteq \mathrm{Ann}_R H^i_{\mathfrak{m}}(M).$$

Since $R/\mathrm{Ann}_R H^i_{\mathfrak{m}}(M)$ is catenary, we have

$$\begin{aligned} & (i - \dim R/\mathfrak{p}) - \dim R_{\mathfrak{p}}/\mathfrak{q}R_{\mathfrak{p}} \\ = \ & (i - \dim R/\mathfrak{p}) - \mathrm{ht}\, \mathfrak{p}/\mathfrak{q} \\ = \ & (i - \dim R/\mathfrak{p}) - (\dim R/\mathfrak{q} - \dim R/\mathfrak{p}) \\ = \ & i - \dim R/\mathfrak{q}. \end{aligned}$$

Hence,

$$\begin{aligned} &\mathfrak{q}R_{\mathfrak{p}} \in \mathrm{Psupp}^{i-\dim R/\mathfrak{p}}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) \\ \iff & H^{(i-\dim R/\mathfrak{p})-\dim R_{\mathfrak{p}}/\mathfrak{q}R_{\mathfrak{p}}}_{\mathfrak{q}R_{\mathfrak{q}}}(M_{\mathfrak{q}}) \neq 0 \\ \iff & H^{i-\dim R/\mathfrak{q}}_{\mathfrak{q}R_{\mathfrak{q}}}(M_{\mathfrak{q}}) \neq 0 \\ \iff & \mathfrak{q} \in \mathrm{Psupp}^i_R(M). \end{aligned}$$

Assume that $R/\mathrm{Ann}_R M$ is catenary. If $\mathfrak{q}R_{\mathfrak{p}} \in \mathrm{Psupp}^{i-\dim R/\mathfrak{p}}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}})$, then

$$H^{(i-\dim R/\mathfrak{p})-\mathrm{ht}\,\mathfrak{p}/\mathfrak{q}}_{\mathfrak{q}R_{\mathfrak{q}}}(M_{\mathfrak{q}}) \neq 0$$

since $(M_{\mathfrak{p}})_{\mathfrak{q}R_{\mathfrak{p}}} \cong M_{\mathfrak{q}}$. It follows that $\mathfrak{q} \in \mathrm{Psupp}^{i+\dim R/\mathfrak{q}-\dim R/\mathfrak{p}-\mathrm{ht}\,\mathfrak{p}/\mathfrak{q}}_R(M)$. Hence, by Lemma 2.2,

$$\mathfrak{p} \supseteq \mathfrak{q} \supseteq \mathrm{Ann}_R H^{i+\dim R/\mathfrak{q}-\dim R/\mathfrak{p}-\mathrm{ht}\,\mathfrak{p}/\mathfrak{q}}_{\mathfrak{m}}(M) \supseteq \mathrm{Ann}_R M.$$

Similar to the first case, we have

$$\mathrm{Psupp}^{i-\dim R/\mathfrak{p}}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) = \{\mathfrak{q}R_{\mathfrak{p}} \mid \mathfrak{q} \in \mathrm{Psupp}^i_R(M),\, \mathfrak{q} \subseteq \mathfrak{p}\}$$

for all $\mathfrak{p} \in \mathrm{Spec}(R)$ and all i. □

Note that the hypothesis of catenaricity of the ring $R/\mathrm{Ann}_R M$ in Lemma 3.1 cannot be omitted. For example, let $(R, \mathfrak{m})$ be a non-catenary Noetherian local domain of dimension 3 (such a domain exists, cf. [1]). Then there exists $\mathfrak{p} \in \mathrm{Spec}(R)$ such that $\dim R/\mathfrak{p} + \mathrm{ht}\,\mathfrak{p} = 2$. So $\dim R/\mathfrak{p} = \mathrm{ht}\,\mathfrak{p} = 1$. Let $\mathfrak{q} = 0$. We can show that $\mathfrak{q}R_{\mathfrak{p}} \in \mathrm{Psupp}^{2-\dim R/\mathfrak{p}}_{R_{\mathfrak{p}}}(R_{\mathfrak{p}})$ but $\mathfrak{q} \notin \mathrm{Psupp}^2_R(R)$.

Proposition 3.2. *Let $i \geq 0$ be an integer. Assume that $R/\mathrm{Ann}_R H^i_{\mathfrak{m}}(M)$ is catenary. Then the following statements are equivalent:*

(i) *$H^i_{\mathfrak{m}}(M)$ satisfies the property $(*)$.*

(ii) *$H^{i-\dim R/\mathfrak{p}}_{\mathfrak{p}R_{\mathfrak{p}}}(M_{\mathfrak{p}})$ satisfies the property $(*)$ for all $\mathfrak{p} \in \mathrm{Supp}(M)$.*

(iii) $\mathrm{Psupp}^i_R(M) = \{\, \widehat{\mathfrak{p}} \cap R \mid \widehat{\mathfrak{p}} \in \mathrm{Psupp}^i_{\widehat{R}}(\widehat{M})\}$.

Proof. (i)⇒(ii) Let $\mathfrak{p} \in \mathrm{Supp}(M)$. By Lemma 2.3(ii), it is enough to prove

$$\mathrm{Psupp}^{i-\dim R/\mathfrak{p}}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) = \mathrm{Var}(\mathrm{Ann}_{R_{\mathfrak{p}}} H^{i-\dim R/\mathfrak{p}}_{\mathfrak{p}R_{\mathfrak{p}}}(M_{\mathfrak{p}})).$$

By Lemma 2.2, we have $\mathrm{Psupp}^{i-\dim R/\mathfrak{p}}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) \subseteq \mathrm{Var}(\mathrm{Ann}_{R_{\mathfrak{p}}} H^{i-\dim R/\mathfrak{p}}_{\mathfrak{p}R_{\mathfrak{p}}}(M_{\mathfrak{p}}))$. Conversely, let $\mathfrak{q}R_{\mathfrak{p}} \in \mathrm{Var}(\mathrm{Ann}_{R_{\mathfrak{p}}} H^{i-\dim R/\mathfrak{p}}_{\mathfrak{p}R_{\mathfrak{p}}}(M_{\mathfrak{p}}))$. Then by Lemma 2.1(i), there exists $\mathfrak{q}'R_{\mathfrak{p}} \in \mathrm{Att}_{R_{\mathfrak{p}}}(H^{i-\dim R/\mathfrak{p}}_{\mathfrak{p}R_{\mathfrak{p}}}(M_{\mathfrak{p}}))$ such that $\mathfrak{q}R_{\mathfrak{p}} \supseteq \mathfrak{q}'R_{\mathfrak{p}}$. So $\mathfrak{q} \supseteq \mathfrak{q}' \in \mathrm{Att}_R(H^i_{\mathfrak{m}}(M))$ by the weak general shifted localization principle. Since $H^i_{\mathfrak{m}}(M)$ satisfies the property $(*)$, it follows by Lemma 2.3(ii) that

$$\mathfrak{q} \in \mathrm{Var}(\mathrm{Ann}_R H^i_{\mathfrak{m}}(M)) = \mathrm{Psupp}^i_R(M).$$

So we have $\mathfrak{q}R_\mathfrak{p} \in \operatorname{Psupp}_{R_\mathfrak{p}}^{i-\dim R/\mathfrak{p}}(M_\mathfrak{p})$ by Lemma 3.1.

(ii)⇒(i) It is trivial.

(i)⇒(iii) If $\mathfrak{p} \in \operatorname{Psupp}_R^i(M)$, then $H_{\mathfrak{p}R_\mathfrak{p}}^{i-\dim R/\mathfrak{p}}(M_\mathfrak{p}) \neq 0$. Let $\widehat{\mathfrak{p}} \in \operatorname{Ass}(\widehat{R}/\mathfrak{p}\widehat{R})$ be such that $\dim \widehat{R}/\widehat{\mathfrak{p}} = \dim R/\mathfrak{p}$. We have $\widehat{\mathfrak{p}} \cap R = \mathfrak{p}$ and the natural homomorphism $R_\mathfrak{p} \to \widehat{R}_{\widehat{\mathfrak{p}}}$ is faithfully flat. Hence, by the flat base change theorem [2, 4.3.2],

$$H_{\widehat{\mathfrak{p}}\widehat{R}_{\widehat{\mathfrak{p}}}}^{i-\dim \widehat{R}/\widehat{\mathfrak{p}}}(\widehat{M}_{\widehat{\mathfrak{p}}}) \cong H_{\mathfrak{p}R_\mathfrak{p}}^{i-\dim R/\mathfrak{p}}(M_\mathfrak{p}) \otimes \widehat{R}_{\widehat{\mathfrak{p}}} \neq 0.$$

So $\widehat{\mathfrak{p}} \in \operatorname{Psupp}_{\widehat{R}}^i(\widehat{M})$. Therefore, $\operatorname{Psupp}_R^i(M) \subseteq \{\widehat{\mathfrak{p}} \cap R \,|\, \widehat{\mathfrak{p}} \in \operatorname{Psupp}_{\widehat{R}}^i(\widehat{M})\}$.

Conversely, let $\widehat{\mathfrak{p}} \in \operatorname{Psupp}_{\widehat{R}}^i(\widehat{M})$. Since $\operatorname{Psupp}_{\widehat{R}}^i(\widehat{M}) = \operatorname{Var}(\operatorname{Ann}_{\widehat{R}} H_\mathfrak{m}^i(M))$, by Lemma 2.1(i), there exists $\widehat{\mathfrak{q}} \in \min \operatorname{Att}_{\widehat{R}}(H_\mathfrak{m}^i(M))$ such that $\widehat{\mathfrak{p}} \supseteq \widehat{\mathfrak{q}}$. So $\widehat{\mathfrak{p}} \cap R \supseteq \widehat{\mathfrak{q}} \cap R \in \operatorname{Att}_R(H_\mathfrak{m}^i(M))$ by Lemma 2.1(ii). Therefore, $\widehat{\mathfrak{p}} \cap R \in \operatorname{Var}(\operatorname{Ann}_R H_\mathfrak{m}^i(M)) = \operatorname{Psupp}_R^i(M)$ by Lemma 2.3(ii).

(iii)⇒(i) By Lemmas 2.3(ii) and 2.2, it is enough to prove

$$\operatorname{Var}(\operatorname{Ann}_R H_\mathfrak{m}^i(M)) \subseteq \operatorname{Psupp}_R^i(M).$$

Let $\mathfrak{p} \in \operatorname{Var}(\operatorname{Ann}_R H_\mathfrak{m}^i(M))$. There exists $\mathfrak{q} \in \min \operatorname{Att}_R(H_\mathfrak{m}^i(M))$ with $\mathfrak{p} \supseteq \mathfrak{q}$ by Lemma 2.1(i). By Lemma 2.1(ii), there exists $\widehat{\mathfrak{q}} \in \operatorname{Att}_{\widehat{R}}(H_\mathfrak{m}^i(M))$ such that $\widehat{\mathfrak{q}} \cap R = \mathfrak{q}$. Since $\operatorname{Psupp}_{\widehat{R}}^i(\widehat{M}) = \operatorname{Var}(\operatorname{Ann}_{\widehat{R}} H_\mathfrak{m}^i(M))$, it holds $\widehat{\mathfrak{q}} \in \operatorname{Psupp}_{\widehat{R}}^i(\widehat{M})$. So by the hypothesis, we have $\mathfrak{q} \in \operatorname{Psupp}_R^i(M)$. Since $R/\operatorname{Ann}_R H_\mathfrak{m}^i(M)$ is catenary, similar to the proof of [3, Lemma 2.2], $\operatorname{Psupp}_R^i(M)$ is closed under specialization. Therefore, $\mathfrak{p} \in \operatorname{Psupp}_R^i(M)$. □

Proof of Theorem 1.1. (i)⇒(iii) By Lemmas 2.3(ii) and 2.2, it is enough to prove

$$\operatorname{Var}(\operatorname{Ann}_R H_\mathfrak{m}^d(M)) \subseteq \operatorname{Psupp}_R^d(M).$$

Let $\mathfrak{p} \in \operatorname{Var}(\operatorname{Ann}_R H_\mathfrak{m}^d(M))$. So $\mathfrak{p} \supseteq \mathfrak{q} \in \min \operatorname{Ann}_R H_\mathfrak{m}^d(M)$. Thus, $\mathfrak{q} \in \operatorname{Att}_R(H_\mathfrak{m}^d(M))$ by Lemma 2.1(i). Then by hypothesis, $\mathfrak{q}R_\mathfrak{p} \in \operatorname{Att}_{R_\mathfrak{p}}(H_{\mathfrak{p}R_\mathfrak{p}}^{d-\dim R/\mathfrak{p}}(M_\mathfrak{p}))$. It follows that $H_{\mathfrak{p}R_\mathfrak{p}}^{d-\dim R/\mathfrak{p}}(M_\mathfrak{p}) \neq 0$. Therefore, $\mathfrak{p} \in \operatorname{Psupp}_R^d(M)$.

(iii)⇔(ii) By Lemma 2.3(i).

(ii)⇒(i) Let $\mathfrak{p} \in \operatorname{Spec}(R)$. We need to prove

$$\operatorname{Att}_{R_\mathfrak{p}}(H_{\mathfrak{p}R_\mathfrak{p}}^{d-\dim R/\mathfrak{p}}(M_\mathfrak{p})) = \{\mathfrak{q}R_\mathfrak{p} \,|\, \mathfrak{q} \in \operatorname{Att}_R(H_\mathfrak{m}^d(M)),\ \mathfrak{q} \subseteq \mathfrak{p}\}.$$

By [2, 11.3.8], it is enough to prove

$$\operatorname{Att}_{R_\mathfrak{p}}(H_{\mathfrak{p}R_\mathfrak{p}}^{d-\dim R/\mathfrak{p}}(M_\mathfrak{p})) \supseteq \{\mathfrak{q}R_\mathfrak{p} \,|\, \mathfrak{q} \in \operatorname{Att}_R(H_\mathfrak{m}^d(M)),\ \mathfrak{q} \subseteq \mathfrak{p}\}.$$

Let $\mathfrak{q} \subseteq \mathfrak{p}$ and $\mathfrak{q} \in \operatorname{Att}_R(H_\mathfrak{m}^d(M))$. By [2, 7.3.2], $\mathfrak{q} \in \operatorname{Ass}_R(M)$ and $\dim R/\mathfrak{q} = d$. It follows that $\mathfrak{q}R_\mathfrak{p} \in \operatorname{Ass}_{R_\mathfrak{p}}(M_\mathfrak{p})$. Since $R/\operatorname{Ann}_R H_\mathfrak{m}^d(M)$ is catenary,

$$\dim R_\mathfrak{p}/\mathfrak{q}R_\mathfrak{p} = \operatorname{ht} \mathfrak{p}/\mathfrak{q} = \dim R/\mathfrak{q} - \dim R/\mathfrak{p} = d - \dim R/\mathfrak{p}.$$

It follows by [2, 11.3.9] that $\mathfrak{q}R_\mathfrak{p} \in \mathrm{Att}_{R_\mathfrak{p}}(H^{d-\dim R/\mathfrak{p}}_{\mathfrak{p}R_\mathfrak{p}}(M_\mathfrak{p}))$.

(iii)$\Leftrightarrow$(v) By Proposition 3.2.

(iv)$\Leftrightarrow$(v) Similar to (i)$\Leftrightarrow$(iii). □

Proof of Theorem 1.2. (i) Assume that $R/\mathrm{Ann}_R M$ is catenary.

(a)$\Rightarrow$(b) Similar to the proof of (i)$\Rightarrow$(iii) of Theorem 1.1.

(b)$\Rightarrow$(a) Let $\mathfrak{q}R_\mathfrak{p} \in \min \mathrm{Att}_{R_\mathfrak{p}}(H^{i-\dim R/\mathfrak{p}}_{\mathfrak{p}R_\mathfrak{p}}(M_\mathfrak{p}))$. Since $H^i_\mathfrak{m}(M)$ satisfies the property $(*)$, so does $H^{i-\dim R/\mathfrak{p}}_{\mathfrak{p}R_\mathfrak{p}}(M_\mathfrak{p})$ by Proposition 3.2. Hence, by Lemma 2.3(ii),

$$\mathrm{Psupp}^{i-\dim R/\mathfrak{p}}_{R_\mathfrak{p}}(M_\mathfrak{p}) = \mathrm{Var}(\mathrm{Ann}_{R_\mathfrak{p}} H^{i-\dim R/\mathfrak{p}}_{\mathfrak{p}R_\mathfrak{p}}(M_\mathfrak{p})).$$

So $\mathfrak{q}R_\mathfrak{p} \in \min \mathrm{Psupp}^{i-\dim R/\mathfrak{p}}_{R_\mathfrak{p}}(M_\mathfrak{p})$ by Lemma 2.1(i). By the weak general shifted localization principle, $\mathfrak{q} \in \mathrm{Att}_R(H^i_\mathfrak{m}(M))$. Let $\mathfrak{q}_1 \in \min \mathrm{Att}_R(H^i_\mathfrak{m}(M))$ be such that $\mathfrak{q} \supseteq \mathfrak{q}_1$. Since $H^i_\mathfrak{m}(M)$ satisfies the property $(*)$, $\mathrm{Psupp}^i_R(M) = \mathrm{Var}(\mathrm{Ann}_R H^i_\mathfrak{m}(M))$ by Lemma 2.3(ii). So $\mathfrak{q}_1 \in \mathrm{Psupp}^i_R(M)$. By Lemma 3.1, $\mathfrak{q}_1 R_\mathfrak{p} \in \mathrm{Psupp}^{i-\dim R/\mathfrak{p}}_{R_\mathfrak{p}}(M_\mathfrak{p})$. Since $\mathfrak{q}R_\mathfrak{p} \supseteq \mathfrak{q}_1 R_\mathfrak{p}$, we get by the minimality of $\mathfrak{q}R_\mathfrak{p}$ that $\mathfrak{q} = \mathfrak{q}_1$. Therefore,

$$\min \mathrm{Att}_{R_\mathfrak{p}}(H^{i-\dim R/\mathfrak{p}}_{\mathfrak{p}R_\mathfrak{p}}(M_\mathfrak{p})) \subseteq \{\mathfrak{q}R_\mathfrak{p} \mid \mathfrak{q} \in \min \mathrm{Att}_R(H^i_\mathfrak{m}(M)),\ \mathfrak{q} \subseteq \mathfrak{p}\}.$$

Conversely, let $\mathfrak{q} \subseteq \mathfrak{p}$ such that $\mathfrak{q} \in \min \mathrm{Att}_R(H^i_\mathfrak{m}(M))$. Then by Lemma 2.1(i), $\mathfrak{q} \in \min \mathrm{Var}(\mathrm{Ann}_R H^i_\mathfrak{m}(M))$ and hence $\mathfrak{q} \in \min \mathrm{Psupp}^i_R(M)$ by Lemma 2.3(ii). By Lemma 3.1, we get $\mathfrak{q}R_\mathfrak{p} \in \mathrm{Psupp}^{i-\dim R/\mathfrak{p}}_{R_\mathfrak{p}}(M_\mathfrak{p})$. Assume

$$\mathfrak{q}R_\mathfrak{p} \supseteq \mathfrak{q}_1 R_\mathfrak{p} \in \min \mathrm{Psupp}^{i-\dim R/\mathfrak{p}}_{R_\mathfrak{p}}(M_\mathfrak{p}).$$

Then $\mathfrak{q} \supseteq \mathfrak{q}_1$ and $\mathfrak{q}_1 \in \mathrm{Psupp}^i_R(M)$ by Lemma 3.1. The minimality of $\mathfrak{q}$ implies $\mathfrak{q} = \mathfrak{q}_1$. Therefore,

$$\mathfrak{q}R_\mathfrak{p} = \mathfrak{q}_1 R_\mathfrak{p} \in \min \mathrm{Psupp}^{i-\dim R/\mathfrak{p}}_{R_\mathfrak{p}}(M_\mathfrak{p}).$$

By Proposition 3.2, $H^{i-\dim R/\mathfrak{p}}_{\mathfrak{p}R_\mathfrak{p}}(M_\mathfrak{p})$ satisfies the property $(*)$, hence by Lemmas 2.1(i) and 2.3(ii), we have

$$\mathfrak{q}R_\mathfrak{p} \in \min \mathrm{Psupp}^{i-\dim R/\mathfrak{p}}_{R_\mathfrak{p}}(M_\mathfrak{p}) = \min \mathrm{Att}_{R_\mathfrak{p}}(H^{i-\dim R/\mathfrak{p}}_{\mathfrak{p}R_\mathfrak{p}}(M_\mathfrak{p})).$$

(b)$\Leftrightarrow$(c) By Proposition 3.2.

(ii) It is obvious that if $H^i_\mathfrak{m}(M)$ satisfies the shifted localization principle, then (c) holds true. On the other hand, we always have (c)$\Rightarrow$(a). Therefore, the statement (ii) is proved. □

It follows by Theorem 1.2(ii) and Lemma 2.3(iv) that if $H^i_\mathfrak{m}(M)$ satisfies the shifted localization principle for all i, then $R/\mathrm{Ann}_R M$ is universally catenary and $R/\mathfrak{p}$ is unmixed for all $\mathfrak{p} \in \mathrm{Ass}_R(M)$. Furthermore, by Lemma 2.3(iii) and Theorem 1.2(i), if $(R, \mathfrak{m})$ is universally catenary and all its formal fibres are Cohen-Macaulay, then $\min \operatorname{Att}_{R_\mathfrak{p}}(H^{i-\dim R/\mathfrak{p}}_{\mathfrak{p}R_\mathfrak{p}}(M_\mathfrak{p})) = \{\mathfrak{q}R_\mathfrak{p} \mid \mathfrak{q} \in \min \operatorname{Att}_R(H^i_\mathfrak{m}(M)),\ \mathfrak{q} \subseteq \mathfrak{p}\}$ for all i and all $\mathfrak{p} \in \operatorname{Spec}(R)$.

Proof of Theorem 1.3. (i) Let $\mathfrak{p} \in \operatorname{Psupp}^i_R(M) \setminus \bigcup_{j=0}^{i-1} \operatorname{Psupp}^j_R(M)$. Then we have $H^{i-\dim R/\mathfrak{p}}_{\mathfrak{p}R_\mathfrak{p}}(M_\mathfrak{p}) \neq 0$ since $\mathfrak{p} \in \operatorname{Psupp}^i_R(M)$. Therefore, $\mathfrak{p} \in \operatorname{Supp}(M)$ and depth $M_\mathfrak{p}$ $\leq i - \dim R/\mathfrak{p}$. Assume

$$\operatorname{depth} M_\mathfrak{p} + \dim R/\mathfrak{p} = j < i.$$

Since depth $M_\mathfrak{p} = j - \dim R/\mathfrak{p}$, it holds $H^{j-\dim R/\mathfrak{p}}_{\mathfrak{p}R_\mathfrak{p}}(M_\mathfrak{p}) \neq 0$. So $\mathfrak{p} \in \operatorname{Psupp}^j_R(M)$. This is a contradiction to the hypothesis. Therefore,

$$\operatorname{Psupp}^i_R(M) \setminus \bigcup_{j=0}^{i-1} \operatorname{Psupp}^j_R(M) \subseteq \{\mathfrak{p} \in \operatorname{Supp}(M) \mid \operatorname{depth} M_\mathfrak{p} + \dim R/\mathfrak{p} = i\}.$$

Conversely, let $\mathfrak{p} \in \operatorname{Supp}(M)$ be such that depth $M_\mathfrak{p} + \dim R/\mathfrak{p} = i$. Hence, $H^{i-\dim R/\mathfrak{p}}_{\mathfrak{p}R_\mathfrak{p}}(M_\mathfrak{p}) \neq 0$. We have $\mathfrak{p} \in \operatorname{Psupp}^i_R(M)$. Assume that there exists j such that $0 \leq j < i$ and $\mathfrak{p} \in \operatorname{Psupp}^j_R(M)$. Then $H^{j-\dim R/\mathfrak{p}}_{\mathfrak{p}R_\mathfrak{p}}(M_\mathfrak{p}) \neq 0$. It follows that

$$\operatorname{depth} M_\mathfrak{p} + \dim R/\mathfrak{p} \leq j < i.$$

This is impossible. So

$$\{\mathfrak{p} \in \operatorname{Supp}(M) \mid \operatorname{depth} M_\mathfrak{p} + \dim R/\mathfrak{p} = i\} \subseteq \operatorname{Psupp}^i_R(M) \setminus \bigcup_{j=0}^{i-1} \operatorname{Psupp}^j_R(M).$$

(ii) Since $H^i_\mathfrak{m}(M)$ satisfies the property $(*)$, $\operatorname{Psupp}^i_R(M) = \operatorname{Var}(\operatorname{Ann}_R H^i_\mathfrak{m}(M))$ by Lemma 2.3(ii). So the proof follows from (i).

(iii) Since $H^j_\mathfrak{m}(M)$ satisfies the property $(*)$ for all $j \leq i$, we have $\operatorname{Psupp}^j_R(M) = \operatorname{Var}(\operatorname{Ann}_R H^j_\mathfrak{m}(M))$ for all $j \leq i$ by Lemma 2.3(ii). Let

$$\mathfrak{p} \in \min \operatorname{Att}_R(H^i_\mathfrak{m}(M)) \setminus \bigcup_{j=0}^{i-1} \operatorname{Var}(\operatorname{Ann}_R H^i_\mathfrak{m}(M)).$$

By (i), we have

$$\mathfrak{p} \in \operatorname{Psupp}^i_R(M) \setminus \bigcup_{j=0}^{i-1} \operatorname{Psupp}^j_R(M) = \{\mathfrak{p} \in \operatorname{Supp}(M) \mid \operatorname{depth} M_\mathfrak{p} + \dim R/\mathfrak{p} = i\}.$$

Assume $\mathfrak{p} \supseteq \mathfrak{q} \in \min\{\mathfrak{p} \in \operatorname{Supp}(M) \mid \operatorname{depth} M_\mathfrak{p} + \dim R/\mathfrak{p} = i\}$. By (i), we have $\mathfrak{q} \in \operatorname{Psupp}^i_R(M)$. On the other hand, $\mathfrak{p} \in \min \operatorname{Att}_R(H^i_\mathfrak{m}(M)) = \min \operatorname{Psupp}^i_R(M)$. Therefore,

$$\mathfrak{p} = \mathfrak{q} \in \min\{\mathfrak{p} \in \operatorname{Supp}(M) \mid \operatorname{depth} M_\mathfrak{p} + \dim R/\mathfrak{p} = i\}.$$

Conversely, let $\mathfrak{p} \in \min\{\mathfrak{p} \in \operatorname{Supp}(M) \mid \operatorname{depth} M_\mathfrak{p} + \dim R/\mathfrak{p} = i\}$. Then by (i) and by the hypothesis,

$$\mathfrak{p} \in \operatorname{Psupp}^i_R(M) \setminus \bigcup_{j=0}^{i-1} \operatorname{Psupp}^j_R(M) = \operatorname{Psupp}^i_R(M) \setminus \bigcup_{j=0}^{i-1} \operatorname{Var}(\operatorname{Ann}_R H^j_\mathfrak{m}(M)).$$

There exists $\mathfrak{q} \in \min \operatorname{Psupp}^i_R(M)$ with $\mathfrak{p} \supseteq \mathfrak{q}$. It is obvious that

$$\mathfrak{q} \notin \bigcup_{j=0}^{i-1} \operatorname{Var}(\operatorname{Ann}_R H^j_\mathfrak{m}(M)) = \bigcup_{j=0}^{i-1} \operatorname{Psupp}^j_R(M).$$

Since $H^i_{\mathfrak{m}}(M)$ satisfies the property $(*)$, by Lemmas 2.3(ii) and 2.1, we have

$$\mathfrak{q} \in \min \operatorname{Psupp}^i_R(M) = \min \operatorname{Att}_R(H^i_{\mathfrak{m}}(M)).$$

Hence, by (ii), $\mathfrak{q}$ belongs to the set $\{\mathfrak{p} \in \operatorname{Att}_R(H^i_{\mathfrak{m}}(M)) \mid \operatorname{depth} M_{\mathfrak{p}} + \dim R/\mathfrak{p} = i\}$. Therefore, $\mathfrak{q} \in \{\mathfrak{p} \in \operatorname{Supp}(M) \mid \operatorname{depth} M_{\mathfrak{p}} + \dim R/\mathfrak{p} = i\}$. The minimality of $\mathfrak{p}$ implies $\mathfrak{p} = \mathfrak{q}$. □

Note that the hypothesis that $H^j_{\mathfrak{m}}(M)$ satisfies the property $(*)$ in Theorem 1.3(ii) cannot be omitted. For example, let $(R, \mathfrak{m})$ be the Noetherian local domain of dimension 2 constructed by [7] such that $\widehat{R}$ has an associated prime ideal $\widehat{\mathfrak{p}}$ of dimension 1. Then $H^1_{\mathfrak{m}}(R)$ does not satisfy the property $(*)$ (cf. [5]) and $\operatorname{Psupp}^0_R(R) = \operatorname{Var}(\operatorname{Ann}_R H^0_{\mathfrak{m}}(R)) = \emptyset$. So 0 belongs to the left-hand side of the formula in (ii) but it does not belong to the right-hand side of this formula. This example also shows that the hypothesis that $H^j_{\mathfrak{m}}(M)$ satisfies the property $(*)$ for all $j \le i$ in Theorem 1.3(iii) cannot be omitted.

The formula in Theorem 1.3(iii) has a connection with Faltings' annihilator theorem. Let $\mathfrak{b} \subseteq \mathfrak{a}$ be ideals of R. In the terminology of Brodmann and Sharp [2], the $\mathfrak{b}$-*finiteness dimension* $f^{\mathfrak{b}}_{\mathfrak{a}}(M)$ *of* M *relative to* $\mathfrak{a}$ is defined as

$$f^{\mathfrak{b}}_{\mathfrak{a}}(M) := \inf\{i \in \mathbb{N} \mid \mathfrak{b} \nsubseteq \sqrt{(0 : H^i_{\mathfrak{a}}(M))}\,\}$$

and the $\mathfrak{b}$-*minimum* $\mathfrak{a}$-*adjusted depth of* M is defined as

$$\lambda^{\mathfrak{b}}_{\mathfrak{a}}(M) = \inf\{\operatorname{depth} M_{\mathfrak{p}} + \operatorname{ht}(\mathfrak{a} + \mathfrak{p})/\mathfrak{p} \mid \mathfrak{p} \in \operatorname{Spec}(R) \backslash \operatorname{Var}(\mathfrak{b})\}.$$

If $\mathfrak{b} = \mathfrak{a}$, we write $f_{\mathfrak{a}}(M)$ instead of $f^{\mathfrak{b}}_{\mathfrak{a}}(M)$, called the *finiteness dimension of* M *relative to* $\mathfrak{a}$. Furthermore, we have (cf. [2, 9.1.2]):

$$\begin{aligned} f_{\mathfrak{a}}(M) &= \inf\{i \in \mathbb{N} \mid H^i_{\mathfrak{a}}(M) \text{ is not finitely generated}\} \\ &= \inf\{i \in \mathbb{N} \mid \mathfrak{a} \nsubseteq \sqrt{(0 : H^i_{\mathfrak{a}}(M))}\,\}. \end{aligned}$$

If R is universally catenary and all its formal fibres are Cohen-Macaulay, then the annihilator theorem holds true over R, i.e., $f^{\mathfrak{b}}_{\mathfrak{a}}(M) = \lambda^{\mathfrak{b}}_{\mathfrak{a}}(M)$ (cf. [2, 9.6.6]). Similar to [2, 11.3.13], we get the following conclusion:

Remark 3.3. Assume that $H^i_{\mathfrak{m}}(M)$ satisfies the property $(*)$ for all i. Let $\mathfrak{b}$ be an ideal of R. Then $f^{\mathfrak{b}}_{\mathfrak{m}}(M) = \lambda^{\mathfrak{b}}_{\mathfrak{m}}(M)$.

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