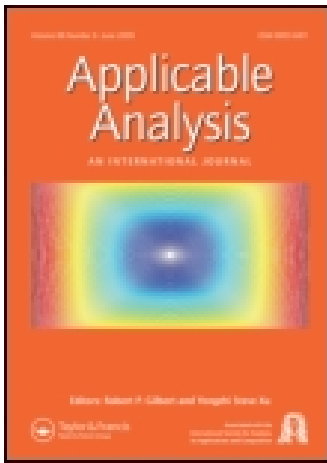


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### Differential stability of convex optimization problems under inclusion constraints

D. T. V. An<sup>a</sup> & N. D. Yen<sup>b</sup>

<sup>a</sup> Department of Mathematics and Informatics, College of Sciences, Thai Nguyen University, Thai Nguyen City, Vietnam.

<sup>b</sup> Institute of Mathematics, Vietnam Academy of Science and Technology, 18 Hoang Quoc Viet, Hanoi 10307, Vietnam.

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## Differential stability of convex optimization problems under inclusion constraints

D.T.V. An<sup>a</sup> and N.D. Yen<sup>b\*</sup>

<sup>a</sup>Department of Mathematics and Informatics, College of Sciences, Thai Nguyen University, Thai Nguyen City, Vietnam; <sup>b</sup>Institute of Mathematics, Vietnam Academy of Science and Technology, 18 Hoang Quoc Viet, Hanoi 10307, Vietnam

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Motivated by the recent work of Mordukhovich et al. [Subgradients of marginal functions in parametric mathematical programming. *Math. Program. Ser. B.* 2009; 116:369–396] on the optimal value function in parametric programming under inclusion constraints, this paper presents some new results on differential stability of convex optimization problems under inclusion constraints and functional constraints in Hausdorff locally convex topological vector spaces. By using the Moreau–Rockafellar theorem and appropriate regularity conditions, we obtain formulas for computing the subdifferential and the singular subdifferential of the optimal value function. By virtue of the convexity, several assumptions used in the above paper by Mordukhovich et al., like the nonemptiness of the Fréchet upper subdifferential of the objective function, the existence of a local upper Lipschitzian selection of the solution map, as well as the  $\mu$ -inner semicontinuity and the  $\mu$ -inner semicompactness of the solution map, are no longer needed. Relationships between our results and the corresponding ones in Aubin’s book [Optima and equilibria. An introduction to nonlinear analysis. 2nd ed. New York (NY): Springer; 1998] are discussed.

**Keywords:** parametric programming under inclusion constraints; convexity; optimal value function; subdifferential; singular subdifferential; the Moreau–Rockafellar theorem; normal cone to the sublevel set of a convex function

**AMS Subject Classifications:** 49J53; 49Q12; 90C25; 90C31

### 1. Introduction

If a mathematical programming problem depends on a parameter, that is, the objective function and the constraints depend on a certain parameter, then the optimal value is a function of the parameter, and the solution map is a set-valued map on the parameter of the problem. In general, the optimal value function is a fairly complicated function of the parameter; it is often nondifferentiable on the parameter, even if the problem in question is a mathematical program with smooth functions on all the variables and on the parameter. This

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\*Corresponding author. Email: [ndyen@math.ac.vn](mailto:ndyen@math.ac.vn)

Dedicated to Professor Boris Sholimovich Mordukhovich on the occasion of his sixty-fifth birthday.

is the reason of the great interest in having formulas for computing generalized directional derivatives (Dini directional derivative, Dini-Hadamard directional derivative, and the Clarke generalized directional derivative) and formulas for evaluating subdifferentials (subdifferential in the sense of convex analysis, Clarke subdifferential, Fréchet subdifferential, limiting subdifferential – also called the Mordukhovich subdifferential) of the optimal value function.

Studies on differentiability properties of the optimal value function and of the solution map in parametric mathematical programming are usually classified as studies on *differential stability* of optimization problems. Aubin [1], Auslender [2], Bonnans and Shapiro [3], Dien and Yen [4], Gauvin and Dubeau [5,6], Gollan [7], Mordukhovich et al. [8], Rockafellar [9], Thibault [10], and many other authors, have had contributions to this research direction.

Motivated by the recent work of Mordukhovich et al. [8] on the optimal value function in parametric programming under inclusion constraints, this paper presents some new results about differential stability of convex optimization problems under inclusion constraints and functional constraints in Hausdorff locally convex topological vector spaces. By using the Moreau–Rockafellar theorem (see e.g. [11, p.48]) and appropriate regularity conditions, we obtain formulas for computing the subdifferential and the singular subdifferential of the optimal value function. By virtue of the convexity, several assumptions used in the above paper by Mordukhovich et al., like the nonemptiness of the Fréchet upper subdifferential of the objective function, the existence of a local upper Lipschitzian selection of the solution map, as well as the  $\mu$ -inner semicontinuity and the  $\mu$ -inner semicompactness of the solution map, are no longer needed. In addition, we can use the Hausdorff locally convex topological vector spaces framework instead of the Banach spaces setting in [8].

Thus, on one hand, our results have the origin in the study of Mordukhovich et al. [8]. On the other hand, they are the results of deepening that study for the case of convex programming problems.

Interestingly, in order to obtain differential stability properties in parametric convex programming, one can use [1] the Fenchel–Moreau theorem [11, p.175]. More precisely, by using that theorem and a series of advanced auxiliary results, Aubin [1, Problem 35 – Subdifferentials of Marginal Functions, p.335] has obtained a formula for computing the subdifferential of the optimal value function under a regularity assumption. This approach requires that the objective function of the problem in question must be convex, *lower semicontinuous*, and the constraint set mapping must be convex and have *closed graph*. The above approach of using the Moreau–Rockafellar theorem does not require the last two additional assumptions on the lower semicontinuity of the objective function and on the closedness of the graph of the constraint set mapping. Therefore, although requiring regularity assumptions which are somewhat stronger than that of Aubin, our results are established for a larger class of convex programming problems, and do not coincide with Aubin’s result if one considers the special case where the spaces are Hilbert and the objective function does not depend on the parameter.

Applied to parametric optimal control problems, with convex objective functions and linear dynamical systems, either discrete or continuous, our results can lead to some rules for the exact computing of the subdifferential and the singular subdifferential of the optimal value function via the data of the given problem.

The organization of the paper is as follows. Section 2 recalls some definitions from variational analysis [12]. Three motivational results from [8] are described in Section 3. Differential stability under convexity, our focus point, is studied in Sections 4 and 5. The

final section compares our results with the results recalled in Section 2 and the above-mentioned result of Aubin.

**2. Preliminaries**

**2.1. Normal cones**

Let  $X$  be a Banach space with the dual denoted by  $X^*$ . For any set-valued map  $F : X \rightrightarrows X^*$ , by

$$\text{Lim sup}_{x \rightarrow \bar{x}} F(x) := \left\{ x^* \in X^* : \exists x_k \rightarrow \bar{x}, x_k^* \xrightarrow{w^*} x^*, \right. \\ \left. x_k^* \in F(x_k) \ \forall k = 1, 2, \dots \right\}$$

one denotes the *sequential Painlevé-Kuratowski upper limit* of  $F$  as  $x$  tends to  $\bar{x}$  with respect to the norm topology of  $X$  and the weak\* topology of  $X^*$ . Here  $x_k^* \xrightarrow{w^*} x^*$  means that the sequence  $\{x_k^*\} \subset X^*$  weakly\* converges to  $x^* \in X^*$ .

If  $\Omega \subset X$  is a given subset, the notation  $x \xrightarrow{\Omega} \bar{x}$  means that  $x \rightarrow \bar{x}$  and  $x \in \Omega$ .

*Definition 2.1* (See [12, Vol. I, p.4]) Let  $\Omega$  be a nonempty subset of  $X$ .

- (i) For any  $x \in \Omega$  and  $\varepsilon \geq 0$ , the set of  $\varepsilon$ -normals of  $\Omega$  at  $x$  is defined by

$$\widehat{N}_\varepsilon(x; \Omega) := \left\{ x^* \in X^* \mid \limsup_{u \xrightarrow{\Omega} x} \frac{\langle x^*, u - x \rangle}{\|u - x\|} \leq \varepsilon \right\}.$$

The set  $\widehat{N}(x; \Omega) := \widehat{N}_0(x; \Omega)$  is called the *Fréchet normal cone* of  $\Omega$  at  $x$ . If  $x \notin \Omega$ , we put  $\widehat{N}_\varepsilon(x; \Omega) = \emptyset$  for all  $\varepsilon \geq 0$ .

- (ii) Let  $\bar{x} \in \Omega$ . The set

$$N(\bar{x}; \Omega) := \text{Lim sup}_{x \rightarrow \bar{x}, \varepsilon \downarrow 0} \widehat{N}_\varepsilon(x; \Omega),$$

is called the *Mordukhovich normal cone* or the *limiting normal cone* of  $\Omega$  at  $\bar{x}$ . We put  $N(\bar{x}; \Omega) = \emptyset$  if  $\bar{x} \notin \Omega$ .

It is clear that  $\widehat{N}(x; \Omega) \subset N(x; \Omega)$  for all  $\Omega \subset X$  and  $x \in \Omega$ . If  $\widehat{N}(x; \Omega) = N(x; \Omega)$  for  $x \in \Omega$ , then one says [12, Def. 1.4] that  $\Omega$  is *normally regular* at  $x$ .

**PROPOSITION 2.1** (See [12, Vol. I, p.6]) *Let  $\Omega$  be a convex set. Then,*

$$\widehat{N}_\varepsilon(\bar{x}; \Omega) = \{x^* \in X^* \mid \langle x^*, x - \bar{x} \rangle \leq \varepsilon \|x - \bar{x}\|, \ \forall x \in \Omega\}$$

for all  $\varepsilon \geq 0$  and  $\bar{x} \in \Omega$ . Especially, the set  $\widehat{N}(\bar{x}; \Omega)$  coincides with the convex cone of  $\Omega$  at  $\bar{x}$  in the sense of convex analysis, that is,

$$\widehat{N}(\bar{x}; \Omega) = \{x^* \in X^* \mid \langle x^*, x - \bar{x} \rangle \leq 0, \ \forall x \in \Omega\}. \tag{2.1}$$

The notions in Definition 2.1 have a local character, since they just depend on the structure of  $\Omega$  in an arbitrarily small neighborhood of the point in question. Thus, one can formulate the results in Proposition 2.1 for locally convex sets, as follows.

PROPOSITION 2.2 (See [12, Vol. I, p.7]) *Let  $\Omega \subset X$  and  $\bar{x} \in \Omega$ . If there is some  $U \in \mathcal{N}(\bar{x})$ , where  $\mathcal{N}(\bar{x})$  denotes the family of the neighborhoods of  $\bar{x}$ , such that  $\Omega \cap U$  is convex, then*

$$\widehat{N}_\varepsilon(\bar{x}; \Omega) = \{x^* \in X^* \mid \langle x^*, x - \bar{x} \rangle \leq \varepsilon \|x - \bar{x}\|, \forall x \in \Omega \cap U\}$$

and

$$N(\bar{x}; \Omega) = \widehat{N}(\bar{x}; \Omega) = \{x^* \in X^* \mid \langle x^*, x - \bar{x} \rangle \leq 0, \forall x \in \Omega \cap U\}.$$

### 2.2. Subdifferentials

Consider a function  $f : X \rightarrow \overline{\mathbb{R}}$  having values in the extended real line  $\overline{\mathbb{R}} = [-\infty, +\infty]$ . One says that  $f$  is *proper* if  $f(x) > -\infty$  for all  $x \in X$ , and the *domain*

$$\text{dom } f := \{x \in X \mid f(x) < \infty\}$$

is nonempty. The *epigraph* and the *hypograph* of  $f$  are given, respectively, by

$$\text{epi } f := \{(x, \alpha) \in X \times \mathbb{R} \mid \alpha \geq f(x)\}$$

and

$$\text{hypo } f := \{(x, \alpha) \in X \times \mathbb{R} \mid \alpha \leq f(x)\}.$$

In the sequel, the notation  $x \xrightarrow{f} \bar{x}$  means that  $x \rightarrow \bar{x}$  and  $f(x) \rightarrow f(\bar{x})$ .

Definition 2.2 (See [12, Vol. I, p.82,83,87]) Let  $f : X \rightarrow \overline{\mathbb{R}}$  be a function defined on a Banach space. Suppose that  $\bar{x} \in X$  and  $|f(\bar{x})| < \infty$ .

- (i) The set

$$\widehat{\partial} f(\bar{x}) := \{x^* \in X^* \mid (x^*, -1) \in \widehat{N}((\bar{x}, f(\bar{x})); \text{epi } f)\}$$

is called the *Fréchet subdifferential* of  $f$  at  $\bar{x}$ .

- (ii) The set

$$\widehat{\partial}^+ f(\bar{x}) := \{x^* \in X^* \mid (-x^*, 1) \in \widehat{N}((\bar{x}, f(\bar{x})); \text{hypo } f)\}$$

is called the *Fréchet upper subdifferential* of  $f$  at  $\bar{x}$ .

- (iii) The set

$$\partial f(\bar{x}) := \{x^* \in X^* \mid (x^*, -1) \in N((\bar{x}, f(\bar{x})); \text{epi } f)\}$$

is said to be the *Mordukhovich subdifferential* or the *limiting subdifferential* of  $f$  at  $\bar{x}$ .

- (iv) The set

$$\partial^\infty f(\bar{x}) := \{x^* \in X^* \mid (x^*, 0) \in N((\bar{x}, f(\bar{x})); \text{epi } f)\}$$

is said to be the *singular subdifferential* of  $f$  at  $\bar{x}$ .

In the case where  $|f(\bar{x})| = \infty$ , one lets  $\widehat{\partial}f(\bar{x})$ ,  $\widehat{\partial}^+f(\bar{x})$ ,  $\partial f(\bar{x})$ , and  $\partial^\infty f(\bar{x})$  to be empty sets.

The inclusion  $\widehat{\partial}f(\bar{x}) \subset \partial f(\bar{x})$  is valid for any  $\bar{x} \in X$ . In the case where  $\widehat{\partial}f(\bar{x}) = \partial f(\bar{x})$  for  $\bar{x} \in \text{dom } f$ , one says [12, Def. 1.91] that  $f$  is *lower regular* at  $\bar{x}$ . If  $f$  is convex, then by [12, Theorem 1.93] we have

$$\widehat{\partial}f(\bar{x}) = \partial f(\bar{x}) = \{x^* \in X^* \mid \langle x^*, x - \bar{x} \rangle \leq f(x) - f(\bar{x}), \forall x \in X\}, \quad (2.2)$$

i.e. the Fréchet subdifferential and the Mordukhovich subdifferential of  $f$  at  $\bar{x}$  coincide with the subdifferential of  $f$  at  $\bar{x}$  in the sense of convex analysis [11, p.196]. In particular,  $f$  is lower regular at  $\bar{x}$ .

Given a subset  $\Omega \subset X$ , one defines the indicator function  $\delta(\cdot; \Omega) : X \rightarrow \overline{\mathbb{R}}$  of  $\Omega$  by setting

$$\delta(x; \Omega) := \begin{cases} 0 & \text{if } x \in \Omega, \\ +\infty & \text{if } x \notin \Omega. \end{cases}$$

PROPOSITION 2.3 (See [12, Vol. I, p.84,88]) *For any  $\bar{x} \in \Omega$ , we have*

$$\widehat{\partial}\delta(\bar{x}; \Omega) = \widehat{N}(\bar{x}; \Omega)$$

and

$$\partial^\infty\delta(\bar{x}; \Omega) = \partial\delta(\bar{x}; \Omega) = N(\bar{x}; \Omega).$$

### 2.3. Coderivatives

Let  $F : X \rightrightarrows Y$  be a set-valued map between Banach spaces. The *graph* and the *domain* of  $F$  are given, respectively, by the formulas

$$\text{gph } F := \{(x, y) \in X \times Y \mid y \in F(x)\},$$

$$\text{dom } F := \{x \in X \mid F(x) \neq \emptyset\}.$$

Equipping the product space  $X \times Y$  with the norm  $\|(x, y)\| := \|x\| + \|y\|$ , by the above notions of normal cones, one can define the concepts of Fréchet coderivative and Mordukhovich coderivative (also called the limiting coderivative) of set-valued maps as follows.

Definition 2.3 (See [12, Vol. I, p.40,41])

- (i) The *Fréchet coderivative* of  $F$  at  $(\bar{x}, \bar{y}) \in \text{gph } F$  is the multifunction  $\widehat{D}^*F(\bar{x}, \bar{y}) : Y^* \rightrightarrows X^*$  given by

$$\widehat{D}^*F(\bar{x}, \bar{y})(y^*) := \{x^* \in X^* \mid (x^*, -y^*) \in \widehat{N}((\bar{x}, \bar{y}); \text{gph } F)\}, \quad \forall y^* \in Y^*. \quad (2.3)$$

- (ii) The *Mordukhovich coderivative* of  $F$  at  $(\bar{x}, \bar{y}) \in \text{gph } F$  is the multifunction  $D^*F(\bar{x}, \bar{y}) : Y^* \rightrightarrows X^*$  given by

$$D^*F(\bar{x}, \bar{y})(y^*) := \{x^* \in X^* \mid (x^*, -y^*) \in N((\bar{x}, \bar{y}); \text{gph } F)\}, \quad \forall y^* \in Y^*.$$

If  $(\bar{x}, \bar{y}) \notin \text{gph } F$  then we accept the convention that the sets  $\widehat{D}^*F(\bar{x}, \bar{y})(y^*)$  and  $D^*F(\bar{x}, \bar{y})(y^*)$  are empty for any  $y^* \in Y^*$ .

**2.4. Optimal value function**

Consider a set-valued map  $G : X \rightrightarrows Y$  between Banach spaces, a function  $\varphi : X \times Y \rightarrow \overline{\mathbb{R}}$ . The *optimal value function* of the *parametric optimization problem under an inclusion constraint*, defined by  $G$  and  $\varphi$ , is the function  $\mu : X \rightarrow \overline{\mathbb{R}}$ , with

$$\mu(x) := \inf \{ \varphi(x, y) \mid y \in G(x) \}. \tag{2.4}$$

By the convention  $\inf \emptyset = +\infty$ , we have  $\mu(x) = +\infty$  for any  $x \notin \text{dom } G$ .

The set-valued map  $G$  (resp., the function  $\varphi$ ) is called the *map describing the constraint set* (resp., the *objective function*) of the problem on the right-hand side of (2.4).

Corresponding to each data pair  $\{G, \varphi\}$  we have one *optimization problem depending on a parameter  $x$* :

$$(P_x) \quad \min \{ \varphi(x, y) \mid y \in G(x) \}. \tag{2.5}$$

Formulas for computing exactly or estimating the Fréchet subdifferential and the Mordukhovich subdifferential of the optimal value function  $\mu(x)$ , to be considered in forthcoming sections, are connected tightly with the *solution map*  $M : \text{dom } G \rightrightarrows Y$ , with

$$M(x) := \{ y \in G(x) \mid \mu(x) = \varphi(x, y) \}, \quad \forall x \in \text{dom } G, \tag{2.6}$$

of the problem  $(P_x)$ .

*Definition 2.4* If  $\text{gph } G$  is a convex set in  $X \times Y$  (that is,  $G$  is a convex set-valued map) and if  $\text{epi } \varphi$  is a convex set in  $X \times Y \times \mathbb{R}$  (that is,  $\varphi$  is a convex function), then (2.5) is said to be a *parametric convex optimization problem*.

It is a simple matter to show that if (2.5) is a convex optimization problem, then  $\mu$  is a convex function. Hence, under that convexity assumption, the Fréchet subdifferential and the Mordukhovich subdifferential of  $\mu$  at  $\bar{x} \in \text{dom } \mu$ , with  $\mu(\bar{x}) \neq -\infty$ , coincide with the subdifferential of  $\mu$  at  $\bar{x}$  in the sense of convex analysis, and these sets can be computed by formula (2.2) with  $f$  being replaced by  $\mu$ .

**3. Motivational results**

**3.1. Fréchet subdifferential of  $\mu(\cdot)$**

The following theorem gives us an upper estimate for the Fréchet subdifferential of the optimal value function in formula (2.4) at a given parameter  $\bar{x}$ . This estimate is established via the Fréchet coderivative of the map  $G$  describing the constraint set and the Fréchet upper subdifferentials of the objective function  $\varphi$ .



**THEOREM 3.1** (See [8, Theorem 1]) *Suppose that the optimal value function  $\mu(\cdot)$  in (2.4) is finite at  $\bar{x} \in \text{dom } M$  and that  $\bar{y} \in M(\bar{x})$  is a vector satisfying  $\widehat{\partial}^+\varphi(\bar{x}, \bar{y}) \neq \emptyset$ . Then,*

$$\widehat{\partial}\mu(\bar{x}) \subset \bigcap_{(x^*, y^*) \in \widehat{\partial}^+\varphi(\bar{x}, \bar{y})} \{x^* + \widehat{D}^*G(\bar{x}, \bar{y})(y^*)\}. \tag{3.1}$$

The estimate (3.1) is valid in the form of an inclusion. It is natural to ask when that inclusion holds as an equality. Additional assumptions based on the following definitions will be made to get the equality.

*Definition 3.1* A map  $h : D \rightarrow Y$  is said to be *locally upper Lipschitzian* at  $\bar{x} \in D$ , where  $D$  is a subset of  $X$ , if there exist  $\eta > 0$  and  $\ell \geq 0$  such that

$$\|h(x) - h(\bar{x})\| \leq \ell \|x - \bar{x}\|, \quad \forall x \in B(\bar{x}, \eta) \cap D.$$

*Definition 3.2* Let  $D \subset X$ . One says that a set-valued map  $F : D \rightrightarrows Y$  has a *local upper Lipschitzian selection* at  $(\bar{x}, \bar{y}) \in \text{gph } F$  if there is a locally upper Lipschitzian single-valued map  $h : D \rightarrow Y$  at  $\bar{x}$  such that  $h(\bar{x}) = \bar{y}$  and  $h(x) \in F(x)$  for all  $x$  belonging to the intersection of  $D$  with a neighborhood of  $\bar{x}$ .

The next theorem gives a sufficient condition for the inclusion (3.1) to hold as equality.

**THEOREM 3.2** (See [8, Theorem 2]) *Suppose that the optimal value function  $\mu(\cdot)$  in (2.4) is finite at  $\bar{x} \in \text{dom } M$ . Assume in addition that  $\varphi$  Fréchet differentiable at  $(\bar{x}, \bar{y})$  and the solution map  $M : \text{dom } G \rightrightarrows Y$  has a local upper Lipschitzian selection at  $(\bar{x}, \bar{y})$ . Then,*

$$\widehat{\partial}\mu(\bar{x}) = x^* + \widehat{D}^*G(\bar{x}, \bar{y})(y^*),$$

with

$$(x^*, y^*) := \nabla\varphi(\bar{x}, \bar{y}) = \left( \frac{\partial\varphi(\bar{x}, \bar{y})}{\partial x}, \frac{\partial\varphi(\bar{x}, \bar{y})}{\partial y} \right)$$

being the gradient of  $\varphi$  at  $(\bar{x}, \bar{y})$ .

The reader is referred to [8] for illustrative examples for Theorems 3.1 and 3.2.

The assumption  $\widehat{\partial}^+\varphi(\bar{x}, \bar{y}) \neq \emptyset$  in Theorem 3.1 is rather strict. It excludes from our consideration convex, Lipschitzian functions of the type  $\varphi(x, y) = |x| + y$ ,  $(x, y) \in \mathbb{R} \times \mathbb{R}$ , or  $\varphi(x, y) = \|x\| + g(y)$ ,  $(x, y) \in X \times Y$ , where  $g : Y \rightarrow \mathbb{R}$  is a given function,  $X$  and  $Y$  are Banach spaces with  $\dim X \geq 1$ . Indeed, for the first example, choosing  $(\bar{x}, \bar{y}) = (0, 0)$  we have  $\widehat{\partial}^+\varphi(\bar{x}, \bar{y}) = \emptyset$ . For the second example, we have  $\widehat{\partial}^+\varphi(\bar{x}, \bar{y}) = \emptyset$  for any  $(\bar{x}, \bar{y}) = (0, v) \in X \times Y$ .

The above remark can be strengthened as follows: *Theorem 3.1 cannot be used for any problem of the form (2.5) with  $\varphi$  being proper, convex, continuous, and nondifferentiable at a given point  $(\bar{x}, \bar{y}) \in \text{gph } M$ .* Indeed, since  $\varphi$  is convex, the Fréchet subdifferential  $\widehat{\partial}\varphi(\bar{x}, \bar{y})$  coincides with the subdifferential in the sense of [11, Subsection 4.2.1]. As  $\varphi$  is continuous at  $(\bar{x}, \bar{y})$ , the latter set is nonempty by [11, Prop. 3, p.199]. Hence, if  $\widehat{\partial}^+\varphi(\bar{x}, \bar{y}) \neq \emptyset$  then  $\varphi$  is Fréchet differentiable at  $(\bar{x}, \bar{y})$  by [12, Prop. 1.87]. This contradicts the condition saying that  $\varphi$  is nondifferentiable at  $(\bar{x}, \bar{y})$ . In the two subsequent sections, we will obtain some results for parametric convex problems of the form (2.5) which allow us to avoid not only

the assumption  $\widehat{\partial}^+ \varphi(\bar{x}, \bar{y}) \neq \emptyset$  in Theorem 3.1, but also the requirement that the solution map admits a local upper Lipschitzian selection in Theorem 3.2.

**3.2. Mordukhovich subdifferential of  $\mu(\cdot)$**

In order to recall a main result from [8] on computing the Mordukhovich subdifferential of  $\mu(\cdot)$ , we have to consider the following definitions.

*Definition 3.3* One says that the solution map  $M(\cdot)$  is  $\mu$ -inner semicontinuous at  $(\bar{x}, \bar{y}) \in \text{gph } M$  if for every sequence  $x_k \xrightarrow{\mu} \bar{x}$  there exists a sequence  $y_k \in M(x_k)$  that contains a subsequence converging to  $\bar{y}$ .

*Definition 3.4* The solution map  $M(\cdot)$  is said to be  $\mu$ -inner semicompact at  $\bar{x}$  if for every sequence  $x_k \xrightarrow{\mu} \bar{x}$  there is a sequence  $y_k \in M(x_k)$  that contains a convergent subsequence.

The properties considered in Definitions 3.3 and 3.4 extend the corresponding notions in [12, Def. 1.63] and adapt them to the solution map  $M(\cdot)$  of (2.5). The only difference is that the condition  $x_k \rightarrow \bar{x}$  in [12] is now replaced by the weaker condition  $x_k \xrightarrow{\mu} \bar{x}$ .

*Definition 3.5* A subset  $\Omega$  in a Banach space  $X$  is called sequentially normally compact (SNC) at  $\bar{x}$  if for any sequences  $\varepsilon_k \downarrow 0$ ,  $x_k \xrightarrow{\Omega} \bar{x}$  and  $x_k^* \in \widehat{N}_{\varepsilon_k}(x_k; \Omega)$  one has

$$[x_k^* \xrightarrow{w^*} 0] \implies [||x_k^*|| \rightarrow 0] \text{ as } k \rightarrow \infty.$$

*Definition 3.6* A set-valued map  $F : X \rightrightarrows Y$  is said to be sequentially normally compact (SNC) at  $(\bar{x}, \bar{y}) \in \text{gph } F$  if its graph possesses this property.

*Definition 3.7* A function  $\varphi : X \rightarrow \overline{\mathbb{R}}$  is called sequentially normally epi-compact (SNEC) at  $\bar{x}$  if its epigraph

$$\text{epi } \varphi := \{(x, \alpha) \in X \times \mathbb{R} \mid \varphi(x) \leq \alpha\}$$

is SNC at  $(\bar{x}, \varphi(\bar{x}))$ .

**THEOREM 3.3** (See [8, Theorem 7]) *Let  $M(\cdot)$  be the solution map defined in (2.6), and let  $\bar{x} \in \text{dom } M$ . The following assertions hold:*

- (i) *For a given vector  $\bar{y} \in M(\bar{x})$ , suppose that  $M$  is  $\mu$ -inner semicontinuous at  $(\bar{x}, \bar{y}) \in \text{gph } M$ , that either  $\varphi$  is SNEC at  $(\bar{x}, \bar{y})$  or  $G$  is SNC at  $(\bar{x}, \bar{y})$ , and the regularity condition*

$$\partial^\infty \varphi(\bar{x}, \bar{y}) \cap (-N((\bar{x}, \bar{y}); \text{gph } G)) = \{0\} \tag{3.2}$$

is satisfied (these assumptions are automatically satisfied if  $\varphi$  is Lipschitz continuous around  $(\bar{x}, \bar{y})$ ). Then one has the inclusions

$$\partial\mu(\bar{x}) \subset \bigcup_{(x^*, y^*) \in \partial\varphi(\bar{x}, \bar{y})} \{x^* + D^*G(\bar{x}, \bar{y})(y^*)\}, \tag{3.3}$$

$$\partial^\infty\mu(\bar{x}) \subset \bigcup_{(x^*, y^*) \in \partial^\infty\varphi(\bar{x}, \bar{y})} \{x^* + D^*G(\bar{x}, \bar{y})(y^*)\}, \tag{3.4}$$

(ii) Assume that  $M$  is  $\mu$ -inner semicompact at  $\bar{x}$  and that the other assumptions of (i) are satisfied at any  $(\bar{x}, \bar{y}) \in \text{gph } M$ . Then one has the inclusions

$$\begin{aligned} \partial\mu(\bar{x}) &\subset \bigcup_{(x^*, y^*) \in \partial\varphi(\bar{x}, \bar{y}), \bar{y} \in M(\bar{x})} \{x^* + D^*G(\bar{x}, \bar{y})(y^*)\}, \\ \partial^\infty\mu(\bar{x}) &\subset \bigcup_{(x^*, y^*) \in \partial^\infty\varphi(\bar{x}, \bar{y}), \bar{y} \in M(\bar{x})} \{x^* + D^*G(\bar{x}, \bar{y})(y^*)\}, \end{aligned}$$

(iii) In addition to the assumptions in (i), assume that  $\varphi$  is strictly differentiable at  $(\bar{x}, \bar{y})$ , that  $M : \text{dom } G \rightrightarrows Y$  admits a local Lipschitzian selection at  $(\bar{x}, \bar{y})$ , and  $G$  is normally regular at  $(\bar{x}, \bar{y})$ . Then the optimal value function  $\mu$  is lower regular at  $\bar{x}$  and (3.3) holds as equality, i.e.

$$\partial\mu(\bar{x}) = \varphi'_x(\bar{x}, \bar{y}) + D^*G(\bar{x}, \bar{y})(\varphi'_y(\bar{x}, \bar{y})). \tag{3.5}$$

We are going to show that if the problem in question is convex then several assumptions in Theorems 3.1–3.3 are no longer needed and, surprisingly, all the upper estimates become equalities.

#### 4. Differential stability under convexity

We now present some new results on differential stability of convex optimization problems under inclusion constraints. By using the Moreau–Rockafellar theorem and appropriate regularity conditions, we will obtain formulas for computing the subdifferential and the singular subdifferential of the optimal value functions.

From now on, if not otherwise stated, we assume that  $X, Y$  are Hausdorff locally convex topological vector spaces with the topological duals denoted, respectively, by  $X^*$  and  $Y^*$ .

For a convex set  $\Omega \subset X$ , the normal cone of  $\Omega$  at  $\bar{x} \in \Omega$  is given by

$$N(\bar{x}; \Omega) = \{x^* \in X^* \mid \langle x^*, x - \bar{x} \rangle \leq 0, \quad \forall x \in \Omega\}. \tag{4.1}$$

As noted in Section 2, formula (4.1) fully agrees with formula (2.1), which was given in a Banach space setting.

For a convex function  $f : X \rightarrow \overline{\mathbb{R}}$ , the subdifferential of  $f$  at

$$\bar{x} \in \text{dom } f = \{x \in X \mid f(x) < \infty\},$$

with  $f(\bar{x}) \neq -\infty$ , is given by

$$\partial f(\bar{x}) = \{x^* \in X^* \mid \langle x^*, x - \bar{x} \rangle \leq f(x) - f(\bar{x}), \quad \forall x \in X\}.$$

This is a generalization of formula (2.2) for the case of functions defined on Hausdorff locally convex topological vector spaces. The set

$$\partial^\infty f(\bar{x}) = \{x^* \in X^* \mid (x^*, 0) \in N((\bar{x}, f(\bar{x})); \text{epi } f)\} \tag{4.2}$$

is called the *singular subdifferential* of  $f$  at  $\bar{x} \in \text{dom } f$ , with  $f(\bar{x}) \neq -\infty$ . We put  $\partial f(\bar{x}) = \emptyset$  and  $\partial^\infty f(\bar{x}) = \emptyset$  if either  $\bar{x} \notin \text{dom } f$  or  $f(\bar{x}) = -\infty$ .

For a convex set-valued map  $F : X \rightrightarrows Y$ , we define the *coderivative* of  $F$  at  $(\bar{x}, \bar{y}) \in \text{gph } F$  as the multifunction  $D^*F(\bar{x}, \bar{y}) : Y^* \rightrightarrows X^*$  given by

$$D^*F(\bar{x}, \bar{y})(y^*) := \{x^* \in X^* \mid (x^*, -y^*) \in N((\bar{x}, \bar{y}); \text{gph } F)\}, \quad \forall y^* \in Y^*$$

with  $N((\bar{x}, \bar{y}); \text{gph } F)$  standing for the normal cone of the convex set  $\text{gph } F \subset X \times Y$  at  $(\bar{x}, \bar{y})$ , i.e.

$$N((\bar{x}, \bar{y}); \text{gph } F) = \{(x^*, y^*) \in X^* \times Y^* \mid \langle x^*, x - \bar{x} \rangle + \langle y^*, y - \bar{y} \rangle \leq 0 \quad \forall (x, y) \in \text{gph } F\}.$$

If  $(\bar{x}, \bar{y}) \notin \text{gph } F$ , then we put  $D^*F(\bar{x}, \bar{y})(y^*) = \emptyset$  for any  $y^* \in Y^*$ .

The following theorem from convex analysis is the main tool in our subsequent proofs.

**THEOREM 4.1** (See [11, Theorem 0.3.3 on p.47–50, Theorem 1 on p.200]) *Let  $f_1, \dots, f_m$  be proper convex functions on  $X$ . Then*

$$\partial(f_1 + \dots + f_m)(x) \supset \partial f_1(x) + \dots + \partial f_m(x)$$

for all  $x \in X$ . If, at a point  $x^0 \in \text{dom } f_1 \cap \dots \cap \text{dom } f_m$ , all the functions  $f_1, \dots, f_m$ , except, possibly, one are continuous, then

$$\partial(f_1 + \dots + f_m)(x) = \partial f_1(x) + \dots + \partial f_m(x)$$

for all  $x \in X$ .

We denote the *interior* of a subset  $\Omega$  of  $X$  by  $\text{int } \Omega$ . Using indicator functions of convex sets, one can easily derive from Theorem 4.1 the next intersection formula.

**PROPOSITION 4.1** (See [11, p.205]) *Let  $A_1, A_2, \dots, A_m$  be convex subsets of  $X$ . Set  $A = A_1 \cap A_2 \cap \dots \cap A_m$  and let  $A_1 \cap (\text{int } A_2) \cap \dots \cap (\text{int } A_m) \neq \emptyset$ . Then,*

$$N(x; A) = N(x; A_1) + N(x; A_2) + \dots + N(x; A_m), \quad \forall x \in X.$$

**PROPOSITION 4.2** *If  $f : X \rightarrow \overline{\mathbb{R}}$  is a convex function, then*

$$\partial^\infty f(x) = N(x; \text{dom } f) \quad \forall x \in X.$$

*Proof* Indeed, since  $\text{epi } f$  is a convex set, by (4.2) we have

$$\begin{aligned} x^* \in \partial^\infty f(x) &\Leftrightarrow (x^*, 0) \in N((x, f(x)); \text{epi } f) \\ &\Leftrightarrow \langle (x^*, 0), (u, \mu) - (x, f(x)) \rangle \leq 0, \quad \forall (u, \mu) \in \text{epi } f \\ &\Leftrightarrow \langle (x^*, 0), (u - x, \mu - f(x)) \rangle \leq 0, \quad \forall (u, \mu) \in \text{epi } f \\ &\Leftrightarrow \langle x^*, u - x \rangle \leq 0, \quad \forall u \in \text{dom } f \\ &\Leftrightarrow x^* \in N(x; \text{dom } f). \end{aligned}$$

This proves that  $\partial^\infty f(x) = N(x; \text{dom } f)$ .  $\square$

Let us turn back our attention to the parametric optimization problem (2.5). The next theorem provides us with formulas for computing the subdifferential and the singular subdifferential of  $\mu$  in the case  $G$  and  $\varphi$  are assumed to be convex.

**THEOREM 4.2** *Let  $G : X \rightrightarrows Y$  be a convex set-valued mapping and  $\varphi : X \times Y \rightarrow \overline{\mathbb{R}}$  be a proper convex function. If at least one of the following regularity conditions is satisfied:*

- (a)  $\text{int}(\text{gph } G) \cap \text{dom } \varphi \neq \emptyset$ ,
- (b)  $\varphi$  is continuous at a point  $(x^0, y^0) \in \text{gph } G$ ,

then for any  $\bar{x} \in \text{dom } \mu$ , with  $\mu(\bar{x}) \neq -\infty$ , and for any  $\bar{y} \in M(\bar{x})$  we have

$$\partial\mu(\bar{x}) = \bigcup_{(x^*, y^*) \in \partial\varphi(\bar{x}, \bar{y})} \{x^* + D^*G(\bar{x}, \bar{y})(y^*)\} \quad (4.3)$$

and

$$\partial^\infty\mu(\bar{x}) = \bigcup_{(x^*, y^*) \in \partial^\infty\varphi(\bar{x}, \bar{y})} \{x^* + D^*G(\bar{x}, \bar{y})(y^*)\}. \quad (4.4)$$

Therefore, if  $M(\bar{x})$  is nonempty then  $\partial\mu(\bar{x})$  and  $\partial^\infty\mu(\bar{x})$  can be computed, respectively, by the formulas (4.3) and (4.4), where the right-hand sides do not depend on the choice of  $\bar{y} \in M(\bar{x})$ .

*Proof* Let  $\bar{x} \in \text{dom } \mu$  and  $\bar{y} \in M(\bar{x})$ . To prove the inclusion “ $\subset$ ” in (4.3), take an arbitrary element  $\bar{x}^* \in \partial\mu(\bar{x})$ . Since the optimal value function  $\mu$  is convex, we have

$$\mu(x) - \mu(\bar{x}) \geq \langle \bar{x}^*, x - \bar{x} \rangle, \quad \forall x \in X.$$

Now, taking an arbitrary  $u \in X$  and selecting a  $v \in G(u)$ , from the above property we get

$$\begin{aligned} \varphi(u, v) - \varphi(\bar{x}, \bar{y}) &= \varphi(u, v) - \mu(\bar{x}) \geq \mu(u) - \mu(\bar{x}) \\ &\geq \langle \bar{x}^*, u - \bar{x} \rangle + \langle 0, v - \bar{y} \rangle. \end{aligned}$$

Therefore,

$$\varphi(u, v) - \varphi(\bar{x}, \bar{y}) \geq \langle (\bar{x}^*, 0), (u, v) - (\bar{x}, \bar{y}) \rangle, \quad \forall (u, v) \in \text{gph } G.$$

Hence

$$\begin{aligned} (\varphi + \delta(\cdot; \text{gph } G))(u, v) - (\varphi + \delta(\cdot; \text{gph } G))(\bar{x}, \bar{y}) \\ \geq \langle (\bar{x}^*, 0), (u, v) - (\bar{x}, \bar{y}) \rangle \quad \forall (u, v) \in X \times Y, \end{aligned} \quad (4.5)$$

where  $\delta((x, y); \text{gph } G) = 0$  if  $(x, y) \in \text{gph } G$ , and  $\delta((x, y); \text{gph } G) = +\infty$  if  $(x, y) \notin \text{gph } G$ , is the indicator function of  $\text{gph } G$ . From (4.5) we have

$$(\bar{x}^*, 0) \in \partial(\varphi + \delta(\cdot; \text{gph } G))(\bar{x}, \bar{y}). \quad (4.6)$$

Since  $\text{gph } G$  is convex,  $\delta(\cdot; \text{gph } G) : X \times Y \rightarrow \overline{\mathbb{R}}$  is convex. Obviously,  $\delta(\cdot; \text{gph } G)$  is continuous at every point belonging to  $\text{int}(\text{gph } G)$ .

Consequently, if the regularity condition (a) is satisfied, then  $\delta(\cdot; \text{gph } G)$  is continuous at a point in  $\text{dom } \varphi$ . By Theorem 4.1, from (4.6) we have

$$\begin{aligned} (\bar{x}^*, 0) &\in \partial\varphi(\bar{x}, \bar{y}) + \partial\delta(\cdot; \text{gph } G)(\bar{x}, \bar{y}) \\ &= \partial\varphi(\bar{x}, \bar{y}) + N((\bar{x}, \bar{y}); \text{gph } G). \end{aligned} \tag{4.7}$$

Thus, there exists  $(x^*, y^*) \in \partial\varphi(\bar{x}, \bar{y})$  such that

$$(\bar{x}^*, 0) \in (x^*, y^*) + N((\bar{x}, \bar{y}); \text{gph } G),$$

or

$$(\bar{x}^* - x^*, -y^*) \in N((\bar{x}, \bar{y}); \text{gph } G),$$

i.e.

$$\bar{x}^* - x^* \in D^*G(\bar{x}, \bar{y})(y^*).$$

The last inclusion implies that

$$\bar{x}^* \in x^* + D^*G(\bar{x}, \bar{y})(y^*). \tag{4.8}$$

Consider the case where the regularity condition (b) is fulfilled. Since

$$\text{dom } \delta(\cdot; \text{gph } G) = \text{gph } G,$$

from (b) it follows that  $\varphi$  is continuous at a point in  $\text{dom } \delta(\cdot; \text{gph } G)$ . Therefore, by Theorem 4.1, from (4.6) we also have (4.7). Thus, there exists  $(x^*, y^*) \in \partial\varphi(\bar{x}, \bar{y})$  such that (4.8) is satisfied.

In both the cases, since  $\bar{x}^* \in \partial\mu(\bar{x})$  can be taken arbitrarily, by (4.8) we can deduce that

$$\partial\mu(\bar{x}) \subset \bigcup_{(x^*, y^*) \in \partial\varphi(\bar{x}, \bar{y})} \{x^* + D^*G(\bar{x}, \bar{y})(y^*)\}.$$

To establish the opposite inclusion, we need to prove that for each element  $(x^*, y^*) \in \partial\varphi(\bar{x}, \bar{y})$ , the following holds true:

$$x^* + D^*G(\bar{x}, \bar{y})(y^*) \subset \partial\mu(\bar{x}).$$

Taking an arbitrary vector  $u^* \in x^* + D^*G(\bar{x}, \bar{y})(y^*)$ , we will show that  $u^* \in \partial\mu(\bar{x})$ . The inclusion  $u^* \in x^* + D^*G(\bar{x}, \bar{y})(y^*)$  yields

$$u^* - x^* \in D^*G(\bar{x}, \bar{y})(y^*). \tag{4.9}$$

Condition (4.9) is equivalent to

$$\begin{aligned} (u^* - x^*, -y^*) &\in N((\bar{x}, \bar{y}); \text{gph } G) \\ \Leftrightarrow (u^* - x^*, -y^*) &\in \partial\delta((\bar{x}, \bar{y}); \text{gph } G) \\ \Leftrightarrow (u^*, 0) &\in (x^*, y^*) + \partial\delta((\bar{x}, \bar{y}); \text{gph } G). \end{aligned}$$

Therefore, we have

$$(u^*, 0) \in \partial\varphi(\bar{x}, \bar{y}) + \partial\delta((\bar{x}, \bar{y}); \text{gph } G).$$

Under each one of the regularity conditions (a) and (b), using Theorem 4.1, from the last inclusion we can deduce that

$$(u^*, 0) \in \partial(\varphi + \delta(\cdot; \text{gph } G))(\bar{x}, \bar{y}).$$

Hence

$$\varphi(x, y) - \varphi(\bar{x}, \bar{y}) \geq \langle u^*, x - \bar{x} \rangle + \langle 0, y - \bar{y} \rangle, \quad \forall (x, y) \in \text{gph } G. \quad (4.10)$$

For each fixed element  $x \in \text{dom } G$ , taking infimum on both sides of (4.10) on  $y \in G(x)$  and remembering that  $\mu(\bar{x}) = \varphi(\bar{x}, \bar{y})$ , we obtain

$$\mu(x) - \mu(\bar{x}) \geq \langle u^*, x - \bar{x} \rangle.$$

Since  $\mu(x) = +\infty$  for all  $x \notin \text{dom } G$ , from the last property it follows that  $u^* \in \partial\mu(\bar{x})$ . Hence (4.3) is valid.

We are going to obtain (4.4) with the aid of some arguments suggested by the anonymous referee of this paper. This new proof is much shorter than our original proof. Observe that  $x \in \text{dom } \mu$  if and only if

$$\mu(x) = \inf\{\varphi(x, y) \mid y \in G(x)\} < \infty.$$

Since the last inequality holds if and only if there exists an  $y \in G(x)$  with  $(x, y) \in \text{dom } \varphi$ , we have

$$\delta(x; \text{dom } \mu) = \inf\{\delta((x, y); \text{dom } \varphi) \mid y \in G(x)\}. \quad (4.11)$$

The representation (4.11) for  $\delta(x; \text{dom } \mu)$  allows us to get (4.4) as a corollary of (4.3). Indeed, since  $\text{dom } \delta(\cdot; \text{dom } \varphi) = \text{dom } \varphi$ , if the regularity requirement in (a) is satisfied then  $\text{int}(\text{gph } G) \cap \text{dom } \delta(\cdot; \text{dom } \varphi) \neq \emptyset$ . Next, if the condition (b) is satisfied then  $(x^0, y^0) \in \text{int}(\text{dom } \varphi)$ ; so  $\delta(\cdot; \text{dom } \varphi)$  is continuous at  $(x^0, y^0) \in \text{gph } G$ . Now, consider the optimization problem (2.5) with  $\varphi(x, y)$  replaced by  $\delta((x, y); \text{dom } \varphi)$ . By (4.11), the corresponding optimal value function  $\mu(x)$  coincides with  $\delta(x; \text{dom } \mu)$ . Therefore, in accordance with (4.3), we have

$$\partial\delta(\cdot; \text{dom } \mu)(\bar{x}) = \bigcup_{(x^*, y^*) \in \partial\delta(\cdot; \text{dom } \varphi)(\bar{x}, \bar{y})} \{x^* + D^*G(\bar{x}, \bar{y})(y^*)\}.$$

The latter yields (4.4) because

$$\partial\delta(\cdot; \text{dom } \mu)(\bar{x}) = N(\bar{x}; \text{dom } \mu) = \partial^\infty\mu(\bar{x})$$

and

$$\partial\delta(\cdot; \text{dom } \varphi)(\bar{x}, \bar{y}) = N((\bar{x}, \bar{y}); \text{dom } \varphi) = \partial^\infty\varphi(\bar{x}, \bar{y})$$

by Proposition 4.2. □

Here are two simple examples designed to illustrate Theorem 4.2.

*Example 4.1* Let  $X = Y = \mathbb{R}$  and  $\bar{x} = 0$ . Consider the optimal value function  $\mu(x)$  in (2.4) with  $\varphi(x, y) = |y|$  and  $G(x) = \left\{y \mid y \geq \frac{1}{2}|x|\right\}$  for all  $x \in \mathbb{R}$ . Then we have  $\mu(x) = \frac{1}{2}|x|$  for all  $x \in \mathbb{R}$ . So  $\partial\mu(\bar{x}) = \left[-\frac{1}{2}, \frac{1}{2}\right]$ ,  $\partial^\infty\mu(\bar{x}) = \{0\}$ , and  $M(\bar{x}) = \{0\}$ . For  $\bar{y} := 0 \in M(\bar{x})$ ,  $\partial\varphi(\bar{x}, \bar{y}) = \{0\} \times [-1, 1]$  and  $\partial^\infty\varphi(\bar{x}, \bar{y}) = \{(0, 0)\}$ . Since  $G$  is a convex set-valued mapping, we have

$N((\bar{x}, \bar{y}); \text{gph } G)$

$$\begin{aligned} &= \left\{ (x^*, y^*) \in \mathbb{R}^2 \mid \langle (x^*, y^*), (x, y) - (0, 0) \rangle \leq 0, \forall (x, y) \in \text{gph } G \right\} \\ &= \left\{ (x^*, y^*) \in \mathbb{R}^2 \mid y^* \leq -2|x^*| \right\} \end{aligned}$$

and

$$D^*G(\bar{x}, \bar{y})(y^*) = \begin{cases} \left[ -\frac{1}{2}y^*, \frac{1}{2}y^* \right] & \text{if } y^* \geq 0, \\ \emptyset & \text{if } y^* < 0. \end{cases}$$

Thus the right-hand sides of (4.3) and of (4.4) can be computed as follows:

$$\begin{aligned} \bigcup_{(x^*, y^*) \in \partial\varphi(\bar{x}, \bar{y})} \{x^* + D^*G(\bar{x}, \bar{y})(y^*)\} &= \bigcup_{y^* \in [-1, 1]} D^*G(\bar{x}, \bar{y})(y^*) \\ &= \bigcup_{y^* \in [-1, 1]} \left[ -\frac{1}{2}y^*, \frac{1}{2}y^* \right] = \left[ -\frac{1}{2}, \frac{1}{2} \right], \\ \bigcup_{(x^*, y^*) \in \partial^\infty\varphi(\bar{x}, \bar{y})} \{x^* + D^*G(\bar{x}, \bar{y})(y^*)\} &= D^*G(\bar{x}, \bar{y})(0) = \{0\}. \end{aligned}$$

As  $\partial\mu(\bar{x}) = [-\frac{1}{2}, \frac{1}{2}]$  and  $\partial^\infty\mu(\bar{x}) = \{0\}$ , the inequalities (4.3) and (4.4) hold.

*Example 4.2* Let  $X = Y = \mathbb{R}$  and  $\bar{x} = 0$ . Let  $\mu(x)$  be defined by (2.4) where  $\varphi(x, y) = |x| + y$  for all  $(x, y) \in \mathbb{R}^2$  and

$$G(x) = \begin{cases} \{y \mid y \geq -\sqrt{x}\} & \text{if } x \geq 0, \\ \emptyset & \text{if } x < 0. \end{cases}$$

We have  $\mu(x) = |x| - \sqrt{x}$  for all  $x \geq 0$ ,  $\mu(x) = +\infty$  for all  $x < 0$ , and  $M(\bar{x}) = \{0\}$ . Hence  $\partial\mu(\bar{x}) = \emptyset$  and  $\partial^\infty\mu(\bar{x}) = (-\infty, 0]$ . For  $\bar{y} := 0 \in M(\bar{x})$ ,  $\partial\varphi(\bar{x}, \bar{y}) = [-1, 1] \times \{1\}$  and  $\partial^\infty\varphi(\bar{x}, \bar{y}) = \{(0, 0)\}$ . By the convexity of  $G$  we have

$N((\bar{x}, \bar{y}); \text{gph } G)$

$$\begin{aligned} &= \left\{ (x^*, y^*) \in \mathbb{R}^2 \mid \langle (x^*, y^*), (x, y) - (0, 1) \rangle \leq 0, \forall (x, y) \in \text{gph } G \right\} \\ &= (-\infty, 0] \times \{0\}; \end{aligned}$$

so  $D^*G(\bar{x}, \bar{y})(0) = (-\infty, 0]$  and  $D^*G(\bar{x}, \bar{y})(y^*) = \emptyset$  for every nonzero  $y^*$ . Then we can calculate the right-hand sides of (4.3) and of (4.4) as follows:

$$\begin{aligned} \bigcup_{(x^*, y^*) \in \partial\varphi(\bar{x}, \bar{y})} \{x^* + D^*G(\bar{x}, \bar{y})(y^*)\} &= \bigcup_{(x^*, y^*) \in [-1, 1] \times \{1\}} \{x^* + D^*G(\bar{x}, \bar{y})(y^*)\} = \emptyset, \\ \bigcup_{(x^*, y^*) \in \partial^\infty\varphi(\bar{x}, \bar{y})} \{x^* + D^*G(\bar{x}, \bar{y})(y^*)\} &= D^*G(\bar{x}, \bar{y})(0) = (-\infty, 0]. \end{aligned}$$

As  $\partial\mu(\bar{x}) = \emptyset$  and  $\partial^\infty\mu(\bar{x}) = (-\infty, 0]$ , the inequalities (4.3) and (4.4) are valid.

### 5. Convex programming problem under functional constraints

We now apply the above general results to convex optimization problems under geometrical and functional constraints. As in the preceding section,  $X$  and  $Y$  are Hausdorff locally convex



topological vector spaces. Consider the problem

$$\min \{ \varphi(x, y) \mid (x, y) \in C, \quad g_i(x, y) \leq 0, \quad i \in I, \quad h_j(x, y) = 0, \quad j \in J \}, \quad (5.1)$$

in which  $\varphi : X \times Y \rightarrow \overline{\mathbb{R}}$  is a convex function,  $C \subset X \times Y$  is a convex set,  $I = \{1, \dots, m\}$ ,  $J = \{1, \dots, k\}$ ,  $g_i : X \times Y \rightarrow \mathbb{R}$  ( $i \in I$ ) are continuous convex functions, and  $h_j : X \times Y \rightarrow \mathbb{R}$  ( $j \in J$ ) are continuous affine functions. For each  $x \in X$ , we put

$$G(x) = \{ y \in Y \mid (x, y) \in C, \quad g_i(x, y) \leq 0, \quad i \in I, \quad h_j(x, y) = 0, \quad j \in J \}. \quad (5.2)$$

It clear that the set-valued map  $G(\cdot)$  given by (5.2) is convex and

$$\text{gph } G = C \cap \left( \bigcap_{i \in I} \Omega_i \right) \cap \left( \bigcap_{j \in J} Q_j \right), \quad (5.3)$$

where  $\Omega_i := \{(x, y) \mid g_i(x, y) \leq 0\}$  ( $i \in I$ ) and  $Q_j := \{(x, y) \mid h_j(x, y) = 0\}$  ( $j \in J$ ) are convex sets.

The following statement is a Farkas lemma for infinite dimensional vector spaces.

**LEMMA 5.1** (See [13, Lemma 1]) *Let  $W$  be a vector space over  $\mathbb{R}$ . Let  $A : W \rightarrow \mathbb{R}^m$  be a linear mapping and  $\gamma : W \rightarrow \mathbb{R}$  be a linear functional. Suppose that  $A$  is represented in the form  $A = (\alpha_i)_i^m$ , where each  $\alpha_i : W \rightarrow \mathbb{R}$  is a linear functional (i.e. for each  $x \in W$ ,  $A(x)$  is a column vector whose  $i$ th component is  $\alpha_i(x)$ , for  $i = 1, \dots, m$ ). Then, the inequality  $\gamma(x) \leq 0$  is a consequence of the inequalities system*

$$\alpha_1(x) \leq 0, \quad \alpha_2(x) \leq 0, \dots, \quad \alpha_m(x) \leq 0$$

if and only if there exist nonnegative real numbers  $\lambda_1, \lambda_2, \dots, \lambda_m \geq 0$  such that

$$\gamma = \lambda_1 \alpha_1 + \dots + \lambda_m \alpha_m.$$

The following lemma describes the normal cone of the intersection of finitely many affine hyperplanes.

**LEMMA 5.2** *Let  $X, Y$  be Hausdorff locally convex topological vector spaces. Let there be given vectors  $(x_j^*, y_j^*) \in X^* \times Y^*$  and real numbers  $\alpha_j \in \mathbb{R}$ ,  $j = 1, \dots, m$ . Set*

$$Q_j = \{(x, y) \mid \langle (x_j^*, y_j^*), (x, y) \rangle = \alpha_j\}.$$

Then, for each  $(\bar{x}, \bar{y}) \in \bigcap_{j=1}^m Q_j$ , we have

$$N \left( (\bar{x}, \bar{y}); \bigcap_{j=1}^m Q_j \right) = \text{span}\{(x_j^*, y_j^*) \mid j = 1, \dots, m\},$$

where  $\text{span}\{(x_j^*, y_j^*) \mid j = 1, \dots, m\}$  denotes the linear subspace generated by the vectors  $(x_j^*, y_j^*)$ ,  $j = 1, \dots, m$ .

We omit the proof of this lemma, because it can be done quite easily by applying Lemma 5.1.

The next lemma will play an important role in the subsequent application of Theorem 4.2 for problem (5.1).

LEMMA 5.3 (See [11, p.206]) *Let  $f$  be a proper convex function on  $X$ , continuous at a point  $x_0 \in X$ . Assume that the inequality  $f(x_1) < f(x_0) = \alpha_0$  holds for some  $x_1 \in X$ . Then,*

$$N(x_0; \mathcal{L}_{\alpha_0} f) = K_{\partial f(x_0)},$$

where  $\mathcal{L}_{\alpha_0} f := \{x \mid f(x) \leq \alpha_0\}$  is a sublevel set of  $f$  and

$$K_{\partial f(x_0)} := \{u^* \in X^* \mid u^* = \lambda x^*, \lambda \geq 0, x^* \in \partial f(x_0)\}$$

is the cone generated by the subdifferential of  $f$  at  $x_0$ .

Let us go back to considering the parametric convex programming problem (5.1). Our first result in this section can be formulated as follows.

THEOREM 5.1 *Suppose that the equality constraints  $h_j(x, y) = 0$  ( $j \in J$ ) are absent in (5.1). If at least one of the following regularity conditions*

- (a1) *There exists a point  $(u^0, v^0) \in \text{dom } \varphi$  such that  $(u^0, v^0) \in \text{int } C$  and  $g_i(u^0, v^0) < 0$  for all  $i \in I$ ,*
- (b1)  *$\varphi$  is continuous at a point  $(x^0, y^0) \in C$  where  $g_i(x^0, y^0) < 0$  for all  $i \in I$  is satisfied, then for any  $\bar{x} \in \text{dom } \mu$ , with  $\mu(\bar{x}) \neq -\infty$ , and for any  $\bar{y} \in M(\bar{x})$  we have*

$$\partial \mu(\bar{x}) = \bigcup_{(x^*, y^*) \in \partial \varphi(\bar{x}, \bar{y})} \{x^* + Q_0^*\} \tag{5.4}$$

and

$$\partial^\infty \mu(\bar{x}) = \bigcup_{(x^*, y^*) \in \partial^\infty \varphi(\bar{x}, \bar{y})} \{x^* + Q_0^*\}, \tag{5.5}$$

where

$$Q_0^* := \left\{ u^* \in X^* \mid (u^*, -y^*) \in N((\bar{x}, \bar{y}); C) + \sum_{i \in I(\bar{x}, \bar{y})} \text{cone } \partial g_i(\bar{x}, \bar{y}) \right\} \tag{5.6}$$

with  $I(\bar{x}, \bar{y}) := \{i \mid g_i(\bar{x}, \bar{y}) = 0\}$  and  $\text{cone } M := \{tz \mid t \geq 0, z \in M\}$  denoting the cone generated by  $M$ .

*Proof* Recall that  $G : X \rightrightarrows Y$ , with  $G(x)$  being defined by (5.2), is a convex set-valued mapping, and the objective function  $\varphi(x, y)$  of (5.1) is convex.

If (a1) is satisfied then it is clear that  $(u^0, v^0) \in \text{int}(\text{gph } G)$ , hence the condition (a) in Theorem 4.2 is fulfilled. If (b1) is satisfied then  $\varphi$  is continuous at the point  $(x^0, y^0)$  which belongs to  $\text{gph } G$ , so the condition (b) in Theorem 4.2 is fulfilled. Therefore, our assumptions guarantee that (4.3) and (4.4) hold.

By the definition of coderivative,

$$D^*G(\bar{x}, \bar{y})(y^*) = \{u^* \in X^* \mid (u^*, -y^*) \in N((\bar{x}, \bar{y}); \text{gph } G)\}. \tag{5.7}$$

Since the constraints  $h_j(x, y) = 0$  ( $j \in J$ ) are absent in (5.1), formula (5.3) becomes

$$\text{gph } G = C \cap \left( \bigcap_{i \in I} \Omega_i \right). \tag{5.8}$$

If (a1) is satisfied, then  $(u^0, v^0) \in (\text{int } C) \cap \left( \bigcap_{i \in I} \text{int } \Omega_i \right)$ . If (b1) is valid, then  $(x^0, y^0) \in C \cap \left( \bigcap_{i \in I} \text{int } \Omega_i \right)$ . So, in both the cases we can use Proposition 4.1 and formula (5.8) to compute the normal cone to  $\text{gph } G$  at  $(\bar{x}, \bar{y})$  as follows

$$N((\bar{x}, \bar{y}); \text{gph } G) = N((\bar{x}, \bar{y}); C) + \sum_{i \in I} N((\bar{x}, \bar{y}); \Omega_i).$$

Since  $N((\bar{x}, \bar{y}); \Omega_i) = \{(0, 0)\}$  for every  $i \notin I(\bar{x}, \bar{y})$ , this formula can be written in the equivalent form

$$N((\bar{x}, \bar{y}); \text{gph } G) = N((\bar{x}, \bar{y}); C) + \sum_{i \in I(\bar{x}, \bar{y})} N((\bar{x}, \bar{y}); \Omega_i). \tag{5.9}$$

By Lemma 5.3, for every  $i \in I(\bar{x}, \bar{y})$  we have

$$N((\bar{x}, \bar{y}); \Omega_i) = K_{\partial g_i(\bar{x}, \bar{y})} = \text{cone } \partial g_i(\bar{x}, \bar{y}).$$

Combining this with (5.6), (5.7), and (5.9), we get (5.4) from (4.3) and (5.5) from (4.4). The proof is complete.  $\square$

We now consider the case where the affine constraints  $h_j(x, y) = 0$  ( $j \in J$ ) are available in (5.1). The second result of this section reads as follows.

**THEOREM 5.2** *For every  $j \in J$ , let  $h_j(x, y) = \langle (x_j^*, y_j^*), (x, y) \rangle - \alpha_j$ , where  $(x_j^*, y_j^*) \in X^* \times Y^*$  and  $\alpha_j \in \mathbb{R}$  ( $j \in J$ ). If  $\varphi$  is continuous at a point  $(x^0, y^0) \in \text{int } C$  with  $g_i(x^0, y^0) < 0$  for all  $i \in I$  and  $h_j(x^0, y^0) = 0$  for all  $j \in J$ , then for any  $\bar{x} \in \text{dom } \mu$ , with  $\mu(\bar{x}) \neq -\infty$ , and for any  $\bar{y} \in M(\bar{x})$  we have*

$$\partial \mu(\bar{x}) = \bigcup_{(x^*, y^*) \in \partial \varphi(\bar{x}, \bar{y})} \{x^* + Q^*\} \tag{5.10}$$

and

$$\partial^\infty \mu(\bar{x}) = \bigcup_{(x^*, y^*) \in \partial^\infty \varphi(\bar{x}, \bar{y})} \{x^* + Q^*\}, \tag{5.11}$$

where

$$Q^* := \left\{ u^* \in X^* \mid (u^*, -y^*) \in N((\bar{x}, \bar{y}); C) + \sum_{i \in I(\bar{x}, \bar{y})} \text{cone } \partial g_i(\bar{x}, \bar{y}) + \text{span}\{(x_j^*, y_j^*), j \in J\} \right\}.$$

*Proof* (This proof follows the same scheme as the proof of Theorem 5.1.) For the set-valued map  $G(\cdot)$  defined by (5.2), we have  $(x^0, y^0) \in \text{gph } G$ . Hence the condition (b) in Theorem 4.2 is satisfied, and we know that (4.3) and (4.4) hold. By our assumptions,

$$(u^0, v^0) \in (\text{int } C) \cap \left( \bigcap_{i \in I} \text{int } \Omega_i \right) \cap \left( \bigcap_{j=1}^k Q_j \right).$$

Therefore, according to Proposition 4.1 and formula (5.3) we have

$$N((\bar{x}, \bar{y}); \text{gph } G) = N((\bar{x}, \bar{y}); C) + \sum_{i \in I} N((\bar{x}, \bar{y}); \Omega_i) + N\left((\bar{x}, \bar{y}); \bigcap_{j=1}^k Q_j\right).$$

Since  $N((\bar{x}, \bar{y}); \bigcap_{j=1}^k Q_j) = \text{span} \left\{ (x_j^*, y_j^*) \mid j \in J \right\}$  by Lemma 5.2,

$$N((\bar{x}, \bar{y}); \Omega_i) = \text{cone } \partial g_i(\bar{x}, \bar{y})$$

for every  $i \in I(\bar{x}, \bar{y})$  by Lemma 5.3, and since  $N((\bar{x}, \bar{y}); \Omega_i) = \{(0, 0)\}$  for every  $i \notin I(\bar{x}, \bar{y})$ , this formula can be written in the form

$$N((\bar{x}, \bar{y}); \text{gph } G) = N((\bar{x}, \bar{y}); C) + \sum_{i \in I(\bar{x}, \bar{y})} N((\bar{x}, \bar{y}); \Omega_i) + \text{span} \left\{ (x_j^*, y_j^*) \mid j \in J \right\}. \tag{5.12}$$

Using (5.7), (5.12), and the definition of  $Q^*$ , we easily get (5.10) from (4.3), (5.11) and from (4.4). □

## 6. Comparisons with some known results

### 6.1. Comparisons with the results of Aubin [1]

Recall that if  $X$  is a Hausdorff locally convex topological vector space,  $f : X \rightarrow \overline{\mathbb{R}}$  is a function having values in the extended real line, then the function  $f^* : X^* \rightarrow \overline{\mathbb{R}}$  given by

$$f^*(x^*) = \sup_{x \in X} [\langle x^*, x \rangle - f(x)], \quad x^* \in X^*, \tag{6.1}$$

is said to be the *conjugate function* of  $f$ . The conjugate function of  $f^*$ , denoted by  $f^{**}$ , is a function defined on  $X$  and having values in  $\overline{\mathbb{R}}$ :

$$f^{**}(x) = \sup_{x^* \in X^*} [\langle x^*, x \rangle - f^*(x^*)] \quad (x \in X).$$

Clearly, the function  $f^{**}$  is convex and closed (in the sense that  $\text{epi } f^{**}$  is closed in the weak topology of  $X \times \mathbb{R}$  or, the same,  $f^{**}$  is lower semicontinuous w.r.t. the weak topology of  $X$ ). According to the Fenchel–Moreau theorem (see [11, Theorem 1, p.175]), if  $f$  is a function on  $X$  everywhere greater than  $-\infty$ , then  $f = f^{**}$  if and only if  $f$  is closed and convex.

By a different approach, Aubin [1, Problem 35 – Subdifferentials of Marginal Functions, p.335] has studied a problem similar to that one considered in the preceding two sections.

Namely, in our notation, Aubin has studied the parametric problem:

$$(P_x) \quad \min\{\varphi(y) \mid y \in G(x)\},$$

where  $X, Y$  are Hilbert spaces,  $\varphi : Y \rightarrow \mathbb{R} \cup \{+\infty\}$  is a proper, convex, *lower semicontinuous* function,  $G : X \rightrightarrows Y$  is convex, *of closed graph*. The optimal value function of that problem is given by

$$\mu(x) = \inf\{\varphi(y) \mid y \in G(x)\}. \quad (6.2)$$

Using the notion of conjugate function, the above Fenchel–Moreau theorem, and some auxiliary results related to continuous linear mappings, convex functions, and convex sets on Hilbert spaces, Aubin has proved the following theorem.

**THEOREM 6.1** (See [1, p.335]) *Assume that*

$$0 \in \text{int}(\text{dom } \varphi - \text{dom } G^{-1}), \quad (6.3)$$

*and that  $\bar{y} \in G(\bar{x})$  is a solution of  $(P_{\bar{x}})$ . Then,  $x^* \in \partial\mu(\bar{x})$  if and only if there exists  $y^* \in \partial\varphi(\bar{y})$  such that*

$$(-y^*, x^*) \in N((\bar{y}, \bar{x}); \text{gph } G^{-1}),$$

*or*

$$(x^*, -y^*) \in N((\bar{x}, \bar{y}); \text{gph } G).$$

*Hence,*

$$\partial\mu(\bar{x}) = D^*G(\bar{x}, \bar{y})(\partial\varphi(\bar{y})).$$

The proof of Aubin is long and rather complicated. The requirements that  $\varphi$  is lower semicontinuous and  $\text{gph } G$  is closed are really needed in Aubin's proof.

The next two claims clarify the connections between the regularity conditions in Theorem 4.2 and the regularity condition in Theorem 6.1.

*Claim 1* *The regularity condition (a) in Theorem 4.2 implies the regularity condition (6.3).*

Indeed, if (a) is fulfilled, then there exist  $(x^0, y^0) \in \text{gph } G$  with  $y^0 \in \text{dom } \varphi$  and  $U \in \mathcal{N}(0_X)$ ,  $V \in \mathcal{N}(0_Y)$ ,  $U$  and  $V$  are open sets, such that

$$(x^0 + U) \times (y^0 + V) \subset \text{gph } G,$$

that is for any  $x \in x^0 + U$  and  $y \in y^0 + V$ ,  $y \in G(x)$ . Hence  $x \in G^{-1}(y)$ , for all  $x \in x^0 + U$  and  $y \in y^0 + V$ . In particular,  $y^0 + V \subset \text{dom } G^{-1}$ . As  $y^0 \in \text{dom } \varphi$ , it follows that

$$0 \in -V = y^0 - (y^0 + V) \subset \text{dom } \varphi - \text{dom } G^{-1}. \quad (6.4)$$

Since  $V$  is open and  $0 \in V$ ,  $-V$  is open and  $0 \in -V$ . Then  $-V \in \mathcal{N}(0)$ . So (6.4) implies that  $0 \in \text{int}(\text{dom } \varphi - \text{dom } G^{-1})$ ; hence (6.3) holds true.

*Claim 2* *The regularity condition (b) in Theorem 4.2 also implies the regularity condition (6.3).*

Indeed, suppose that  $\varphi$  is continuous at such a point  $y^0$  that there is  $x^0 \in X$  with  $(x^0, y^0) \in \text{gph } G$ . Then, for every  $\varepsilon > 0$ , there exists  $V \in \mathcal{N}(0)$  such that

$$|\varphi(y^0 + v) - \varphi(y^0)| < \varepsilon, \quad \forall v \in V.$$

It follows that  $y^0 + V \subset \text{dom } \varphi$ . Since  $y^0 \in G(x^0)$ ,  $x^0 \in G^{-1}(y^0)$ . So we have

$$0 \in V = (y^0 + V) - y^0 \subset \text{dom } \varphi - \text{dom } G^{-1};$$

thus (6.3) is valid.

## 6.2. Comparisons with the results of Mordukhovich et al. [8]

The assertions of Theorem 4.2 are similar to those of the three theorems of Mordukhovich et al. [8] which have been recalled in Section 3. By imposing the strong convexity requirement on (2.5), we need not to rely on the assumption  $\widehat{\partial}^+ \varphi(\bar{x}, \bar{y}) \neq \emptyset$  in Theorem 3.1, the condition saying that the solution map  $M : \text{dom } G \rightrightarrows Y$  has a local upper Lipschitzian selection at  $(\bar{x}, \bar{y})$  in Theorem 3.2, as well as the  $\mu$ -inner semicontinuity and the  $\mu$ -inner semicompactness conditions on the solution map  $M(\cdot)$ , in Theorem 3.3.

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## References

- [1] Aubin J-P. Optima and equilibria. An introduction to nonlinear analysis. 2nd ed. New York (NY): Springer; 1998.
- [2] Auslender A. Differential stability in nonconvex and nondifferentiable programming. Math. Program. Study. 1979;10:29–41.
- [3] Bonnans JF, Shapiro A. Perturbation analysis of optimization problems. New York (NY): Springer; 2000.
- [4] Dien PH, Yen ND. On implicit function theorems for set-valued maps and their application to mathematical programming under inclusion constraints. Appl. Math. Optim. 1991;24:35–54.
- [5] Gauvin J, Dubeau F. Differential properties of the marginal function in mathematical programming. Math. Program. Study. 1982;19:101–119.
- [6] Gauvin J, Dubeau F. Some examples and counterexamples for the stability analysis of nonlinear programming problems. Math. Program. Study. 1984;21:69–78.

- [7] Gollan B. On the marginal function in nonlinear programming. *Math. Oper. Res.* 1984;9:208–221.
- [8] Mordukhovich BS, Nam NM, Yen ND. Subgradients of marginal functions in parametric mathematical programming. *Math. Program. Ser. B.* 2009;116:369–396.
- [9] Rockafellar RT. Lagrange multipliers and subderivatives of optimal value functions in nonlinear programming. *Math. Program. Study.* 1982;17:28–66.
- [10] Thibault L. On subdifferentials of optimal value functions. *SIAM J. Control Optim.* 1991;29:1019–1036.
- [11] Ioffe AD, Tihomirov VM. *Theory of extremal problems.* Amsterdam: North-Holland; 1979.
- [12] Mordukhovich BS. *Variational analysis and generalized differentiation, Volume I: Basic theory, Volume II: Applications.* Berlin: Springer; 2006.
- [13] Bartl D. A short algebraic proof of the Farkas lemma. *SIAM J. Optim.* 2008;19:234–239.