AN IMPLICIT ITERATION METHOD FOR VARIATIONAL INEQUAITIES OVER THE SET OF COMMON FIXED POINTS FOR A FINITE FAMILY OF NONEXPANSIVE MAPPINGS IN HILBERT SPACES

Nguyen Buong[1] and Nguyen Thi Quynh Anh[2]
[1]Vietnamese Academy of Science and Technology
Institute of Information Technology
18, Hoang Quoc Viet, Cau Giay, Ha Noi, Viet Nam
E-mail: nbuong@ioit.ac.vn
[2] Department of Basic sciences-Thai Nguyen university of Information and com
Thai Nguyen National University, Vietnam
E-mail: namlinhtn@gmail.com

October 20, 2013

Abstract

In this paper, we introduce a new implicit iteration method for finding a solution for a variational inequality involving Lipschitz continuous and strongly monotone mapping over the set of common fixed points for a finite family of nonexpansive mappings on Hilbert spaces.

1 2

1. INTRODUCTION

Let C be a nonempty closed and convex subset of a real Hilbert space H with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$ and let $F : H \to H$ be a nonlinear mapping. The variational inequality problem is formulated as finding a point $p^* \in C$ such that

$$\langle F(p^*), p - p^* \rangle \geq 0, \quad \forall p \in C. \tag{1.1}$$

[1]Key words: Contraction, common fixed points, and nonexpansive mappings.
[2]AMS 2000 Mathematics Subject Classification (MSC): 41A65, 47H17, 47H20.

Variational inequalities were initially studied by Stampacchia in [1] and ever since have been widely investigated, since they cover as diverse disciplines as partial differential equations, optimal control, optimization, mathematical programming, mechanics, and finance (see, [1]-[3]).

It is well known that, if F is a L-Lipschitz continuous and η-strongly monotone, i.e., F satisfies the following conditions:

$$\|F(x) - F(y)\| \leq L\|x - y\|;$$
$$\langle F(x) - F(y), x - y \rangle \geq \eta\|x - y\|^2,$$

where L and η are fixed positive numbers, then (1.1) has a unique solution. It is also known that (1.1) is equivalent to the fixed-point equation

$$p = P_C(p - \mu F(p)), \tag{1.2}$$

where P_C denotes the metric projection from $x \in H$ onto C and μ is an arbitrarily fixed positive constant.

Let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive self-mappings of C. For finding an element $p \in \cap_{i=1}^N Fix(T_i)$, Xu and Ori introduced in [4] the following implicit iteration process. For $x_0 \in C$ and $\{\beta_k\}_{k=1}^\infty \subset (0, 1)$, the sequence $\{x_k\}$ is generated as follows:

$$\begin{aligned}
&x_1 = \beta_1 x_0 + (1 - \beta_1)T_1 x_1,\\
&x_2 = \beta_2 x_1 + (1 - \beta_2)T_2 x_2,\\
&\cdots\cdots\cdots\cdots\cdots\cdots\\
&x_N = \beta_N x_{N-1} + (1 - \beta_N)T_N x_N,\\
&x_{N+1} = \beta_{N+1} x_N + (1 - \beta_{N+1})T_1 x_{N+1},\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned}$$

The compact expression of the method is the form

$$x_k = \beta_k x_{k-1} + (1 - \beta_k)T_{[k]} x_k, \quad k \geq 1, \tag{1.3}$$

where $T_{[n]} = T_{n \ modN}$, for integer $n \geq 1$, with the mod function taking values in the set $\{1, 2, \cdots, N.\}$ They proved the following result.

Theorem 1.1. *Let H be a real Hilbert space and C be a nonempty closed convex subset of H. Let $\{T_i\}_{i=1}^N$ be N nonexpansive self-maps of C such that $\cap_{i=1}^N Fix(T_i) \neq \emptyset$, where $Fix(T_i) = \{x \in C : T_i x = x\}$. Let $x_0 \in C$ and $\{\beta_k\}_{k=1}^\infty$ be a sequence in $(0, 1)$ such that $\lim_{k\to\infty} \beta_k = 0$. Then, the sequence $\{x_k\}$ defined implicitly by (1.3) converges weakly to a common fixed point of the mappings $\{T_i\}_{i=1}^N$.*

Further, Zeng and Yao introduced in [5] the following implicit method. For an arbitrary initial point $x_0 \in H$, the sequence $\{x_k\}_{k=1}^{\infty}$ is generated as follows:

$$
\begin{aligned}
&x_1 = \beta_1 x_0 + (1-\beta_1)[T_1 x_1 - \lambda_1 \mu F(T_1 x_1)],\\
&x_2 = \beta_2 x_1 + (1-\beta_2)[T_2 x_2 - \lambda_2 \mu F(T_2 x_2)],\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&x_N = \beta_N x_{N-1} + (1-\beta_N)[T_N x_N - \lambda_N \mu F(T_N x_N)],\\
&x_{N+1} = \beta_{N+1} x_N + (1-\beta_{N+1})[T_1 x_{N+1} - \lambda_{N+1} \mu F(T_1 x_{N+1})],\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned}
$$

The scheme is written in a compact form as

$$x_k = \beta_k x_{k-1} + (1-\beta_k)[T_{[k]} x_k - \lambda_k \mu F(T_{[k]} x_k)], \quad k \geq 1. \tag{1.4}$$

They proved the following result.

Theorem 1.2. *Let H be a real Hilbert space and $F : H \to H$ be a mapping such that for some constants $L, \eta > 0$, F is L-Lipschitz continuous and η-strongly monotone. Let $\{T_i\}_{i=1}^{N}$ be N nonexpansive self-maps of H such that $C = \cap_{i=1}^{N} Fix(T_i) \neq \emptyset$. Let $\mu \in (0, 2\eta/L^2)$, let $x_0 \in H, \{\lambda_k\}_{k=1}^{\infty} \subset [0,1)$ and $\{\beta_k\}_{k=1}^{\infty} \subset (0,1)$ satisfying the conditions: $\sum_{k=1}^{\infty} \lambda_k < \infty$ and $\alpha \leq \beta_k \leq \beta, k \geq 1$, for some $\alpha, \beta \in (0,1)$. Then, the sequence $\{x_k\}$ defined by (1.4) converges weakly to a common fixed point of the mappings $\{T_i\}_{i=1}^{N}$. The convergence is strong if and only if $\liminf_{k\to\infty} d(x_k, C) = 0$.*

Clearly, from $\sum_{k=1}^{\infty} \lambda_k < \infty$ we have that $\lambda_k \to 0$ as $k \to \infty$. To obtain strong convergence without the condition $\sum_{k=1}^{\infty} \lambda_k < \infty$, in this paper we propose the following implicit algorithm:

$$x_t = T^t x_t, \quad T^t := T_0^t T_N^t ... T_1^t, \quad t \in (0,1), \tag{1.5}$$

where T_i^t are defined by

$$T_i^t x = (1-\beta_t^i)x + \beta_t^i T_i x, \quad i = 1, \cdots, N, \quad T_0^t y = (I - \lambda_t \mu F)y, \ x, y \in H, \tag{1.6}$$

I denotes the identity mapping of H, and the parameters $\{\lambda_t\}, \{\beta_t^i\} \subset (0,1)$ for all $t \in (0,1)$ satisfy the following conditions: $\lambda_t \to 0$ as $t \to 0$ and $0 < \liminf_{t\to 0} \beta_t^i \leq \limsup_{t\to 0} \beta_t^i < 1, i = 1, \cdots, N$.

2. MAIN RESULT

We formulate the following facts for the proof of our results.

Lemma 2.1 [6]. (i) $\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y\rangle$ *and for any fixed* $t \in [0,1]$
(ii) $\|(1-t)x+ty\|^2 = (1-t)\|x\|^2 + t\|y\|^2 - (1-t)t\|x-y\|^2, \quad \forall x, y \in H.$

With $T = I$, from [7], we have the following fact.

Lemma 2.2. $\|T_0^t x - T_0^t y\| \leq (1-\lambda_t\tau)\|x-y\|, \quad \forall x, y \in H$ *and for a fixed number* $\mu \in (0, 2\eta/L^2)$, *where* $\tau = 1 - \sqrt{1-\mu(2\eta-\mu L^2)} \in (0,1)$.

Lemma 2.3(*Demiclosedness Principle* [8]). *Assume that* T *is a nonexpansive self-mapping of a closed convex subset* K *of a Hibert space* H. *If* T *has a fixed point, then* $I - T$ *is demiclosed; that is, whenever* $\{x_k\}$ *is a sequence in* K *weakly converging to some* $x \in K$ *and the sequence* $\{(I-T)x_k\}$ *strongly converges to some* y, *it follows that* $(I-T)x = y$.

Now, we are in a position to prove the following result.

Theorem 2.4. *Let* H *be a real Hilbert space and* $F : H \to H$ *be a mapping such that for some constants* $L, \eta > 0$, F *is L-Lipschitz continuous and* η*-strongly monotone. Let* $\{T_i\}_{i=1}^N$ *be* N *nonexpansive self-maps of* H *such that* $C = \cap_{i=1}^N Fix(T_i) \neq \emptyset$. *Let* $\mu \in (0, 2\eta/L^2)$ *and let* $t \in (0,1), \{\lambda_t\}, \{\beta_t^i\} \subset (0,1)$, *such that*

$$\lambda_t \to 0, as \quad t \to 0 \quad and \quad 0 < \liminf_{t\to 0} \beta_t^i \leq \limsup_{t\to 0} \beta_t^i < 1, \quad i = 1, \cdots, N.$$

Then, the net $\{x_t\}$ *defined by (1.5)-(1.6) converges strongly to the unique element* p^* *in (1.1).*

Proof. By Lemma 2.2, we have that

$$\begin{aligned}
\|T^t x - T^t y\| &\leq (1-\lambda_t\tau)\|T_N^t ... T_1^t x - T_N^t ... T_1^t y\| \\
&\cdots\cdots\cdots\cdots\cdots\cdots \\
&\leq (1-\lambda_t\tau)\|T_i^t ... T_1^t x - T_i^t ... T_1^t y\| \\
&\cdots\cdots\cdots\cdots\cdots\cdots \\
&\leq (1-\lambda_t\tau)\|T_1^t x - T_1^t y\| \leq (1-\lambda_t\tau)\|x-y\| \quad \forall x, y \in H.
\end{aligned}$$

So, T^t is a contraction in H. By Banach's Contraction Principle, there exists a unique element $x_t \in H$ such that $x_t = T^t x_t$ for all $t \in (0,1)$.

Next, we show that $\{x_t\}$ is bounded. Indeed, for a fixed point $p \in C$, we have that $T_i^t p = p$ for $i = 1, \cdots, N$, and hence

$$\begin{aligned}
\|x_t - p\| &= \|T^t x_t - p\| = \|T^t x_t - T_N^t ... T_1^t p\| \\
&= \|(I-\lambda_t\mu F)T_N^t ... T_1^t x_t - (I-\lambda_t\mu F)T_N^t ... T_1^t p - \lambda_t\mu F(p)\| \\
&\leq (1-\lambda_t\tau)\|T_N^t ... T_1^t x_t - T_N^t ... T_1^t p\| + \lambda_t\mu\|F(p)\| \\
&\leq (1-\lambda_t\tau)\|T_{N-1}^t ... T_1^t x_t - T_{N-1}^t ... T_1^t p\| + \lambda_t\mu\|F(p)\| \\
&\cdots\cdots\cdots\cdots\cdots\cdots \\
&\leq (1-\lambda_t\tau)\|T_i^t ... T_1^t x_t - T_i^t ... T_1^t p\| + \lambda_t\mu\|F(p)\|
\end{aligned}$$

$$\begin{aligned}&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\&\le (1-\lambda_t\tau)\|T_1^t x_t - T_1^t p\| + \lambda_t\mu\|F(p)\|\\&\le (1-\lambda_t\tau)\|x_t - p\| + \lambda_t\mu\|F(p)\|.\end{aligned}$$

Therefore,

$$\|x_t - p\| \le \frac{\mu}{\tau}\|F(p)\|$$

that implies the boundedness of $\{x_t\}$. So, are the nets $\{F(y_t^N)\}, \{y_t^i\}, i = 1, \cdots, N$.

Put

$$\begin{aligned}y_t^1 &= (1-\beta_t^1)x_t + \beta_t^1 T_1 x_t,\\y_t^2 &= (1-\beta_t^2)y_t^1 + \beta_t^2 T_2 y_t^1,\\&\cdots\cdots\cdots\cdots\cdots\cdots\\y_t^i &= (1-\beta_t^i)y_t^{i-1} + \beta_t^i T_i y_t^{i-1},\\&\cdots\cdots\cdots\cdots\cdots\cdots\\y_t^N &= (1-\beta_t^N)y_t^{N-1} + \beta_t^N T_N y_t^{N-1}.\end{aligned} \tag{2.1}$$

Then,

$$x_t = (I - \lambda_t\mu F)y_t^N. \tag{2.2}$$

Moreover,

$$\begin{aligned}\|x_t - p\|^2 &= \|(I-\lambda_t\mu F)y_t^N - p\|^2\\&= \|y_t^N - p\|^2 - 2\lambda_t\mu\langle F(y_t^N), y_t^N - p\rangle + \lambda_t^2\mu^2\|F(y_t^N)\|^2\\&\le \|y_t^{N-1} - p\|^2 - 2\lambda_t\mu\langle F(y_t^N), y_t^N - p\rangle + \lambda_t^2\mu^2\|F(y_t^N)\|^2\\&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\&\le \|y_t^1 - p\|^2 - 2\lambda_t\mu\langle F(y_t^N), y_t^N - p\rangle + \lambda_t^2\mu^2\|F(y_t^N)\|^2\\&\le \|x_t - p\|^2 - 2\lambda_t\mu\langle F(y_t^N), y_t^N - p\rangle + \lambda_t^2\mu^2\|F(y_t^N)\|^2\end{aligned}$$

Thus,

$$\eta\|y_t^N - p\|^2 + \langle F(p), y_t^N - p\rangle \le \frac{\lambda_t\mu}{2}\|F(y_t^N)\|^2. \tag{2.3}$$

Further, for the sake of simplicity, we put $y_t^0 = x_t$ and prove that

$$\|y_t^i - T_i y_t^{i-1}\| \to 0,$$

as $t \to 0$ for $i = 1, \cdots, N$.

Let $\{t_k\} \subset (0,1)$ be an arbitrary sequence converging to zero as $k \to \infty$ and $x_k := x_{t_k}$. We have to prove that $\|y_k^i - T_i y_k^{i-1}\| \to 0$, where y_k^i are defined by (2.1) with $t = t_k$ and $y_k^i = y_{t_k}^i$. Let $\{x_l\}$ be a subsequence of $\{x_k\}$ such that

$$\limsup_{k\to\infty}\|y_k - T_i y_k^{i-1}\| = \lim_{l\to\infty}\|y_l^i - T_i y_l^{i-1}\|.$$

Let $\{x_{k_j}\}$ be a subsequence of $\{x_l\}$ such that

$$\limsup_{k\to\infty}\|x_k - p\| = \lim_{j\to\infty}\|x_{k_j} - p\|.$$

From (2.2) and Lemma 2.1, it implies that

$$\begin{aligned}
\|x_{k_j} - p\|^2 &= \|(I - \lambda_{k_j}\mu F)y_{k_j}^N - p\|^2 \\
&\leq \|y_{k_j}^N - p\|^2 - 2\lambda_{k_j}\mu\langle F(y_{k_j}^N), x_{k_j} - p\rangle \\
&= \|(1-\beta_{k_j}^N)(y_{k_j}^{N-1} - p) + \beta_{k_j}^N(T_N y_{k_j}^{N-1} - T_N p)\|^2 \\
&\quad - 2\lambda_{k_j}\mu\langle F(y_{k_j}^N), x_{k_j} - p\rangle \\
&\leq (1-\beta_{k_j}^N)\|y_{k_j}^{N-1} - p\|^2 + \beta_{k_j}^N\|T_N y_{k_j}^{N-1} - T_N p\|^2 \\
&\quad - 2\lambda_{k_j}\mu\langle F(y_{k_j}^N), x_{k_j} - p\rangle \\
&\leq \|y_{k_j}^{N-1} - p\|^2 - 2\lambda_{k_j}\mu\langle F(y_{k_j}^N), x_{k_j} - p\rangle \\
&\leq \cdots \leq \|y_{k_j}^1 - p\|^2 - 2\lambda_{k_j}\mu\langle F(y_{k_j}^N), x_{k_j} - p\rangle \\
&\leq \|x_{k_j} - p\|^2 - 2\lambda_{k_j}\mu\langle F(y_{k_j}^N), x_{k_j} - p\rangle.
\end{aligned}$$

Hence,

$$\lim_{j\to\infty}\|x_{k_j} - p\| = \lim_{j\to\infty}\|y_{k_j}^i - p\|, \quad i = 1, \cdots, N. \tag{2.4}$$

By Lemma 2.1,

$$\begin{aligned}
\|y_{k_j}^i - p\|^2 &= (1-\beta_{k_j}^i)\|y_{k_j}^{i-1} - p\|^2 + \beta_{k_j}^i\|T_i y_{k_j}^{i-1} - p\|^2 \\
&\quad - \beta_{k_j}^i(1-\beta_{k_j}^i)\|y_{k_j}^i - T_i y_{k_j}^{i-1}\|^2 \\
&\leq (1-\beta_{k_j}^i)\|y_{k_j}^{i-1} - p\|^2 + \beta_{k_j}^i\|y_{k_j}^{i-1} - p\|^2 \\
&\quad - \beta_{k_j}^i(1-\beta_{k_j}^i)\|y_{k_j}^i - T_i y_{k_j}^{i-1}\|^2 \\
&= \|y_{k_j}^{i-1} - p\|^2 - \beta_{k_j}^i(1-\beta_{k_j}^i)\|y_{k_j}^i - T_i y_{k_j}^{i-1}\|^2 \\
&\leq \cdots = \|y_{k_j}^0 - p\|^2 - \beta_{k_j}^i(1-\beta_{k_j}^i)\|y_{k_j}^i - T_i y_{k_j}^{i-1}\|^2 \\
&= \|x_{k_j} - p\|^2 - \beta_{k_j}^i(1-\beta_{k_j}^i)\|y_{k_j}^i - T_i y_{k_j}^{i-1}\|^2, \quad i = 1, \cdots, N.
\end{aligned}$$

Without loss of generality, we can assume that $\alpha \leq \beta_t^i \leq \beta$ for some $\alpha, \beta \in (0,1)$. Then, we have

$$\alpha(1-\beta)\|y_{k_j}^i - T_i y_{k_j}^{i-1}\|^2 \leq \|x_{k_j} - p\|^2 - \|y_{k_j}^i - p\|^2.$$

This together with (2.4) implies that

$$\lim_{j\to\infty}\|y_{k_j}^i - T_i y_{k_j}^{i-1}\|^2 = 0, \quad i = 1, \cdots, N.$$

It means that $\|y_t^i - T_i y_t^{i-1}\| \to 0$ as $t \to 0$ for $i = 1, \cdots, N$. On the other hand, from

$$\|y_t^i - T_i y_t^{i-1}\| = (1 - \beta_t^i)\|y_t^{i-1} - T_i y_t^{i-1}\|,$$

which is followed from (2.1), and $0 < \alpha \leq \beta_t^i \leq \beta < 1$, it follows that $\|y_t^{i-1} - T_i y_t^{i-1}\| \to 0$ as $t \to 0$.

Next, we show that $\|x_t - T_i x_t\| \to 0$ as $t \to 0$. In fact, in the case that $i = 1$ we have $y_t^0 = x_t$. So, $\|x_t - T_1 x_t\| \to 0$ as $t \to 0$. Further, since

$$\|y_t^1 - T_1 x_t\| = (1 - \beta_t^1)\|x_t - T_1 x_t\|$$

and $\|x_t - T_1 x_t\| \to 0$, we have that $\|y_t^1 - T_1 x_t\| \to 0$. Therefore, from

$$\|x_t - y_t^1\| \leq \|x_t - T_1 x_t\| + \|T_1 x_t - y_t^1\|$$

it follows that $\|x_t - y_t^1\| \to 0$ as $t \to 0$. On the other hand, since

$$\|y_t^2 - T_2 y_t^1\| = (1 - \beta_t^2)\|y_t^1 - T_2 y_t^1\| \to 0$$

and

$$\begin{aligned}\|y_t^2 - x_t\| &\leq (1 - \beta_t^2)\|y_t^1 - x_t\| + \beta_t^2\|T_2 y_t^1 - x_t\| \\ &\leq (1 - \beta_t^2)\|y_t^1 - x_t\| + \beta_t^2\|T_2 y_t^1 - y_t^1\| + \|y_t^1 - x_t\|\end{aligned}$$

we obtain that $\|y_t^2 - x_t\| \to 0$ as $t \to 0$. Now, from

$$\begin{aligned}\|x_t - T_2 x_t\| &\leq \|x_t - y_t^2\| + \|y_t^2 - T_2 y_t^1\| + \|T_2 y_t^1 - T_2 x_t\| \\ &\leq \|x_t - y_t^2\| + \|y_t^2 - T_2 y_t^1\| + \|y_t^1 - x_t\|\end{aligned}$$

and $\|x_t - y_t^2\|, \|y_t^2 - T_2 y_t^1\|, \|y_t^1 - x_t\| \to 0$, it follows that $\|x_t - T_2 x_t\| \to 0$. Similarly, we obtain that $\|x_t - T_i x_t\| \to 0$, for $i = 1, \cdots, N$ and $\|y_t^N - x_t\| \to 0$ as $t \to 0$.

Let $\{x_k\}$ be any sequence of $\{x_t\}$ converging weakly to $\tilde{p}$ as $k \to \infty$. Then, $\|x_k - T_i x_k\| \to 0$, for $i = 1, \cdots, N$ and $\{y_k^N\}$ also converges weakly to $\tilde{p}$. By Lemma 2.3, we have $\tilde{p} \in C = \cap_{i=1}^N Fix(T_i)$ and from (2.3), it follows that

$$\langle F(p), p - \tilde{p}\rangle \geq 0 \quad \forall p \in C.$$

Since $p, \tilde{p} \in C$, by replacing p by $tp + (1-t)\tilde{p}$ in the last inequality, dividing by t and taking $t \to 0$ in the just obtained inequality, we obtain

$$\langle F(\tilde{p}), p - \tilde{p}\rangle \geq 0 \quad \forall p \in C.$$

The uniqueness of p^* in (1.1) guarantees that $\tilde{p} = p^*$. Again, replacing p in (2.3) by p^*, we obtain the strong convergence for $\{x_t\}$. This completes the proof.

3. APPLICATION

Recall that a mapping $S : H \to H$ is called a γ-strictly pseudocontractive, if there exists a constant $\gamma \in [0, 1)$ such that

$$\|Sx - Sy\|^2 \le \|x - y\|^2 + \gamma\|(I - S)x - (I - S)y\|^2, \quad \forall x, y \in H.$$

It is well-known [9] that a mapping $T : H \to H$ by $Tx = \alpha x + (1 - \alpha)Sx$ with a fixed $\alpha \in [\gamma, 1)$ for all $x \in H$ is a nonexpansive mapping and $Fix(T) = Fix(S)$. Using this fact, we can extend our result to the case $C = \cap_{i=1}^N Fix(S_i)$, where S_i is γ_i-strictly pseudocontractive as follows.

Let $\alpha_i \in [\gamma_i, 1)$ be fixed numbers. Then, $C = \cap_{i=1}^N Fix(\tilde{T}_i)$ with $\tilde{T}_i y = \alpha_i y + (1 - \alpha_i)S_i y$, a nonexpansive mapping, for $i = 1 \cdot\cdot, N$, and hence

$$\begin{aligned}\tilde{T}_i^t y &= (1 - \beta_t^i)y + \beta_t^i \tilde{T}_i y \\ &= (1 - \beta_t^i(1 - \alpha_i))y + \beta_t^i(1 - \alpha_i)S_i y, \quad i = 1 \cdot\cdot, N.\end{aligned} \tag{3.1}$$

So, we have the following result.

Theorem 3.1. *Let H be a real Hilbert space and $F : H \to H$ be a mapping such that for some constants $L, \eta > 0$, F is L-Lipschitz continuous and η-strongly monotone. Let $\{S_i\}_{i=1}^N$ be N γ_i-strictly pseudocontractive self-maps of H such that $C = \cap_{i=1}^N Fix(S_i) \ne \emptyset$. Let $\alpha_i \in [\gamma_i, 1), \mu \in (0, 2\eta/L^2)$ and let $t \in (0, 1), \{\lambda_t\}, \{\beta_t^i\} \subset (0, 1)$, such that*

$$\lambda_t \to 0, as \quad t \to 0 \quad and \quad 0 < \liminf_{t\to 0} \beta_t^i \le \limsup_{t\to 0} \beta_t^i < 1, \quad i = 1, \cdot\cdot\cdot, N.$$

Then, the net $\{x_t\}$ defined by

$$x_t = \tilde{T}^t x_t, \quad \tilde{T}^t := T_0^t \tilde{T}_N^t ... \tilde{T}_1^t, \quad t \in (0, 1),$$

where $\tilde{T}_i^t$, for $i = 1, \cdot\cdot\cdot, N$, are defined by (3.1) and $T_0^t x = (I - \lambda_t \mu F)x$, converges strongly to the unique element p^* in (1.1).

It is known in [10] that $Fix(\tilde{S}) = C$ where $\tilde{S} = \sum_{i=1}^N \xi_i S_i$ with $\xi_i > 0$ and $\sum_{i=1}^N \xi_i = 1$ for N γ_i-strictly pseudocontractions $\{S_i\}_{i=1}^N$. Moreover, $\tilde{S}$ is γ-strictly pseudocontractive with $\gamma = \max\{\gamma_i : 1 \le i \le N\}$. So, we also have the following result.

Theorem 3.2. *Let H be a real Hilbert space and $F : H \to H$ be a mapping such that for some constants $L, \eta > 0$, F is L-Lipschitz continuous and η-strongly monotone. Let $\{S_i\}_{i=1}^N$ be N γ_i-strictly pseudocontractive self-maps of H such that $C = \cap_{i=1}^N Fix(S_i) \ne \emptyset$. Let $\alpha \in [\gamma, 1)$, where $\gamma = \max\{\gamma_i : 1 \le i \le N\}$, $\mu \in (0, 2\eta/L^2)$ and let $t \in (0, 1), \{\lambda_t\}, \{\beta_t\} \subset (0, 1)$, such that*

$$\lambda_t \to 0, as \quad t \to 0 \quad and \quad 0 < \liminf_{t\to 0} \beta_t \le \limsup_{t\to 0} \beta_t < 1.$$

Then, the net $\{x_t\}$*, defined by*

$$x_t = \tilde{T}^t x_t, \quad \tilde{T}^t := T_0^t((1-\beta_t(1-\alpha))I + \beta_t(1-\alpha)\sum_{i=1}^{N}\xi_i S_i), \quad t \in (0,1),$$

where $T_0^t = (I - \lambda_t \mu F)$*,* $\xi_i > 0$ *and* $\sum_{i=1}^N \xi_i = 1$*, converges strongly to the unique element* p^* *in (1.1).*

This work was supported by the Vietnamese National Foundation of Science and Technology Development.

References

[1] D. Kinderlehrer, and G. Stampacchia, An introduction to variational inequalities and their applications, Academic Press, New York, NY, 1980.

[2] R. Glowinski, Numerical methods for nonlinear variational problems, Springer, New York, NY 1984.

[3] E. Zeidler, Nonlinear functional analysis and its applications, Springer, New York, NY 1985.

[4] H.K. Xu and R.G. Ori, An implicit iteration process for nonexpansive mappings, Numer. Func. Anal. Optim. 22 (2001)767-773.

[5] L.C. Zeng and J.Ch. Yao, Implicit iteration scheme with perturbed mapping for common fixed points of a finite family of nonexpansive mappings, Nonl. Anal. 64 (2006) 2507-2515.

[6] G. Marino and H.K. Xu, Weak and strong convergence theorems for strict pseudo-contractions in Hibert spaces, J. Math. Anal. Appl. 329 (2007) 336-346.

[7] Y. Yamada, The hybrid steepest-descent method for variational inequalities problems over the intesectionof the fixed point sets of nonexpansive mappings, Inhently parallel algorithms in feasibility and optimization and their applications, Edited by D. Butnariu, Y. Censor, and S. Reich, North-Holland, Amsterdam, Holland, pp. 473-504, 2001.

[8] K. Goebel and W.A. Kirk, Topics in Metric Fixed Point Theory, Cambridge Studies in Advanced Math., V. 28, Cambridge Univ. Press, Cambridge 1990.

[9] H. Zhou, Convergence theorems of fixed points for k-strict pseudo-contractions in Hibert spaces, Nonl. Anal. 69 (2008) 456-462.

[10] G.L. Aced and H.K Xu, Iterative method for strict pseudocontractions in Hibert spaces, Nonlinear Anal. 67 (2007) 2258-2271.