ON THE EXISTENCE OF SOLUTIONS TO MIXED PARETO QUASI-OPTIMIZATION PROBLEMS

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Abstract. In this paper, we introduce mixed Pareto quasi-optimization problems and show some sufficient conditions on the existence of their solutions. As special cases, we obtain several results for the mixed Pareto quasi-equilibrium problem and also system of two Pareto quasi-optimization problems.

Key Words. *Mixed Pareto quasi-optimization problems, C-convex, C-quasiconvex-like mappings, C- continuous mappings, diagonally C_2-convex, diagonally C_2-quasi-convex-like mappings.*

1 Introduction

Throughout this paper, unless otherwise specify, we denote by X, Y, Y_1, Y_2, Z real locally convex Hausdorff topological vector spaces. Assume that $D \subset X, K \subset Z$ are nonempty subsets. and $C_i \subseteq Y_i, i = 1, 2$, are convex closed cones. 2^A denotes the collection of all subsets in the set A. Given multivalued mappings $S : D \times K \to 2^D, T : D \times K \to 2^K; P : D \to 2^D, Q : K \times D \to 2^K$ and single-valued mappings $F_1 : K \times K \times D \to Y_1, F_2 : K \times D \times D \to Y_2$, we consider the following problem:

Mixed Pareto quasi-optimization problems

Find $(\bar{x}, \bar{y}) \in D \times K$

$\bar{x} \in S(\bar{x}, \bar{y}); \bar{y} \in T(\bar{x}, \bar{y})$ such that

there are no $v \in T(\bar{x}, \bar{y}), v \neq \bar{y}, t \in P(\bar{x})$,

$y \in Q(\bar{x}, t), t \neq \bar{x}$ with

$F_1(\bar{y}, \bar{y}, \bar{x}) \succeq_{C_1} F_1(\bar{y}, v, \bar{x})$;

$F_2(y, \bar{x}, \bar{x}) \succeq_{C_2} F_2(y, \bar{x}, t)$.

Where $a \succeq_C b$ means that $a - b \in C$.

The multivalued mapping $Q(x, .) : D \to 2^K$, $\mathrm{Gr}Q(\bar{x}, .) = \{(t, y) \in D \times K | y \in Q(\bar{x}, t)\}$ Setting $(\mathrm{Gr}Q(\bar{x}, .)) \cap (P(\bar{x}) \times K) = A \times B$, the above problem becomes to find $(\bar{x}, \bar{y}) \in D \times K$ such that

$$\bar{x} \in S(\bar{x}, \bar{y}); \bar{y} \in T(\bar{x}, \bar{y});$$
$$F_1(\bar{y}, \bar{y}, \bar{x}) \in PMin(F_1(\bar{y}, T(\bar{x}, \bar{y}), \bar{x}) | C_1);$$
$$F_2(y, \bar{x}, \bar{x}) \in PMin(F_2(A, \bar{x}, B) | C_2),$$

with $PMin(A|C) = \{x \in A |$ there are no $y \in A, y \neq x$ such that $x \succeq_C y\}$ is the set of Pareto efficient points of A to C.

The purpose of this paper is to study the existence for solutions of mixed Pareto quasi-optimization problems and its applications to different problems.

2 Preliminaries

Let Y be a Hausdorff locally convex topological vector spaces and let $C \subseteq Y$ be a cone. We denote $l(C) = C \cap (-C)$. If $l(C) = \{0\}$, C is said to be pointed. Let Y' be the topological dual space of Y. We denote by $\langle \xi, y \rangle$ the value of $\xi \in Y'$ at $y \in Y$. The topological dual cone C', strict topological dual cone C'^{+} of C are defined as

$$C' = \{\xi \in Y' : \langle \xi, c \rangle \geq 0, \text{ for all } c \in C\},$$
$$C'^{+} = \{\xi \in Y' : \langle \xi, c \rangle > 0, \text{ for all } c \in C \setminus l(C)\}.$$

In this paper, we allways assume that C is a pointed cone in Y and $C'^{+} \neq \emptyset$.

The following concept (see in [1], [3], [5] and [6]) is used in our studies.

Definition 2.1 $A/$
1) $F : D \to 2^Y$ is said to be *upper (lower) C–continuous* in $\bar{x} \in \mathrm{dom}\, F$ if for any neighborhood V of the origin in Y there is a neighborhood U of $\bar{x}$ such that $F(x) \subset F(\bar{x}) + V + C$ ($F(\bar{x}) \subset F(x) + V - C$, respectively)
holds for all $x \in U \cap \mathrm{dom} F$.
2) F is *upper (lower) C-convex on D* if for any

$x_1,\, x_2 \in D,\, \alpha \in [0,1]$, it holds $\alpha F(x_1) + (1-\alpha)F(x_2) \subseteq F(\alpha x_1 + (1-\alpha)x_2) + C$ (respectively, $F(\alpha x_1 + (1-\alpha)x_2) \subseteq \alpha F(x_1) + (1-\alpha)F(x_2) - C)$).

3) F is upper (lower) C-quasi-convex-like on D if for any $x_1,\, x_2 \in D,\, \alpha \in [0,1]$, either $F(x_1) \subseteq F(\alpha x_1 + (1-\alpha)x_2) + C$ or, $F(x_2) \subseteq F(\alpha x_1 + (1-\alpha)x_2) + C$ (respectively, either $F(\alpha x_1 + (1-\alpha)x_2) \subseteq F(x_1) - C$ or, $F(\alpha x_1 + (1-\alpha)x_2) \subseteq F(x_2) - C)$, holds.

B/

(i) $F : D \times D \to 2^Y$ is called *diagonally upper (lower) C-convex in the second variable* if for any finite set $\{x_1, ..., x_n\} \subseteq D, x \in co\{x_1, ..., x_n\}, x = \sum\limits_{j=1}^{n} \alpha_j x_j, \alpha_j \geq 0, \sum\limits_{j=1}^{n} \alpha_j = 1$, it holds

$\sum\limits_{j=1}^{n} \alpha_j F(x, x_j) \subseteq F(x,x) + C$

(respectively, $F(x,x) \subseteq \sum\limits_{j=1}^{n} \alpha_j F(x, x_j) - C$).

(ii) F is called *diagonally upper (lower) C-quasi-convex-like* in the second variable if for any finite set $\{x_1, ..., x_n\} \subseteq D, x \in co\{x_1, ..., x_n\}, x = \sum\limits_{j=1}^{n} \alpha_j x_j, \alpha_j \geq 0, \sum\limits_{j=1}^{n} \alpha_j = 1$, there is an index $j \in \{1, ..., n\}$ it holds $F(x, x_j) \subseteq F(x,x) + C$, (respectively, $F(x,x) \subseteq F(x, x_j) - C$).

We need the following lemmas in the sequel.

Lemma 2.2 ([1]) *Let X, D and Y be as in the above lemma, $C \subseteq Y$ be a cone and $\xi \in C'$, $F : D \to Y$ be a lower C -continuous mapping. Then the function $f : D \to \mathbb{R}$ defined by*

$$f(x) = \langle \xi, F(x) \rangle,$$

is a lower semicontinuous function.

Lemma 2.3 ([1]) *Let X, D and Y be as in the above lemma, $C \subseteq Y$ be a cone and $\xi \in C'$, $F : D \to Y$ be a C -convex mapping. Then the function $g : D \to \mathbb{R}$ defined by*

$$g(x) = \langle \xi, F(x) \rangle,$$

is a convex function.

The proofs are trivial.

In order to prove the main theorem, we use the following lemma (see Lemma 4.2 in [7]).

Lemma 2.4 *Let D, K be nonempty compact convex subsets of locally convex Hausdorff topological vector spaces X, Y, respectively. Given multivalued mappings $S : D \times K \to 2^D, H : D \times K \to 2^K; M : D \to 2^D$. We suppose that:*

(i) S is a multivalued with nonempty convex values and has open lower sections;

(ii) H is upper semi-continuous with nonempty closed convex values and the set $A = \{(x,y) \mid x \in S(x,y), y \in H(x,y)\}$ is closed;

(iii) M has open lower sections and for all $x \in D, x \notin coM(x)$.

Then there exists $(\bar{x}, \bar{y}) \in D \times K$ with $\bar{x} \in S(\bar{x}, \bar{y}), \bar{y} \in H(\bar{x}, \bar{y})$ and $S(\bar{x}, \bar{y}) \cap M(\bar{x}) = \emptyset$.

3 Existence of solutions

Given multivalued mappings S, T, P, Q and $F_i, i = 1, 2$ with nonempty values as in Introduction, we prove the following results:

Theorem 3.1 *We assume that the following conditions hold:*

(i) D, K are nonempty convex compact subsets;

(ii) S is a multivalued with nonempty convex values and has open lower sections and T is a continuous multivalued mapping with nonempty closed convex values and the subset $A = \{(x,y) \in D \times K | (x,y) \in S(x,y) \times T(x,y)\}$ is closed;

(iii) P has open lower sections and $P(x) \subseteq S(x,y)$ for $(x,y) \in A$. For any fixed $t \in D$, the multivalued mapping $Q(.,t) : D \to 2^K$ is lower semi-continuous with compact values;

(iv) The mapping F_1 is a $(-C_1)-$ continuous and C_1- continuous mapping. For any fixed $t \in D$ the mapping $F_2(.,.,t) : K \times D \to Y_2$ is a $(-C_2)$- continuous mapping and for any fixed $y \in K$, the mapping $N_2 : K \times D \to Y_2$ defined by $N_2(y,x) = F_2(y,x,x)$ is C_2-continuous ;

(v) For any fixed $(x,y) \in D \times K$, the mapping $F_1(y,.,x) : K \to Y_1$ is C_1- convex (or, C_1-quasi-convex-like) and any $y \in K$ the mapping $F_2(y,.,.) : D \times D \to Y_2$ is diagonally C_2-convex in the second variable (or, diagonally C_2-quasi-convex-like in the second variable).

Then there exists $(\bar{x},\bar{y}) \in D \times K$ such that

$$\begin{aligned} &\bar{x} \in S(\bar{x},\bar{y}); \bar{y} \in T(\bar{x},\bar{y}) \textit{ such that} \\ &\textit{there are no } v \in T(\bar{x},\bar{y}), v \neq \bar{y}, t \in P(\bar{x}), \\ &\qquad y \in Q(\bar{x},t), t \neq \bar{x} \textit{ with} \\ &F_1(\bar{y},\bar{y},\bar{x}) \succeq_{C_1} F_1(\bar{y},v,\bar{x}); \\ &F_2(y,\bar{x},\bar{x}) \succeq_{C_2} F_2(y,\bar{x},t). \end{aligned}$$

Proof. Let $\xi_i \in C_i'^{+}, i = 1,2$ be fixed. Let $\epsilon > 0$ be arbitrary. Since ξ_i is continuous, there exists a neighborhood V of the origin in Y such that $\xi_i(V) \subseteq (-\frac{\epsilon}{2}, \frac{\epsilon}{2})$. We define the multivalued mapping $H : D \times K \to 2^K$ by $H(x,y) = \{y' \in T(x,y) : \langle \xi_1, F_1(y,y',x)\rangle \leq \langle \xi_1, F_1(y,v,x)\rangle, \forall v \in T(x,y)\}$.

For each $(x,y) \in D \times K$, we first show that $H(x,y)$ is a nonempty set. Indeed, for each $(x,y) \in D \times K, T(x,y)$ is a compact subset. Therefore, we apply Lemma 2.2 in Section 2 with $D = T(x,y)$ and $F = F_1(y,.,x)$ to conclude that the function $f : T(x,y) \to \mathbb{R}$ defined by $f(v) = \langle \xi_1, F_1(y,v,x)\rangle$, is a lower semicontinuous function and hence there is $y' \in T(x,y)$ such that $f(y') = \min\limits_{v \in T(x,y)} f(v)$. This shows $\langle \xi_1, F_1(y,y',x)\rangle \leq \langle \xi_1, F_1(y,v,x)\rangle, \forall v \in T(x,y)$ and so $y' \in H(x,y)$.

Further, we show that $H(x,y)$ is convex set, for all $(x,y) \in D \times K$. Indeed, let $y_1', y_2' \in H(x,y)$ and $\lambda \in [0,1]$. The convexity of $T(x,y)$ yields $\lambda y_1' + (1-\lambda)y_2' \in T(x,y)$ and

$$\begin{aligned} \langle \xi_1, F_1(y,y_1',x)\rangle &\leq \langle \xi_1, F_1(y,v,x)\rangle, \\ \langle \xi_1, F_1(y,y_2',x)\rangle &\leq \langle \xi_1, F_1(y,v,x)\rangle, \forall v \in T(x,y). \end{aligned}$$

If $F_1(y,.,x)$ is C_1-convex, we then have $F_1(y, \lambda y_1' + (1-\lambda)y_2', x) \preceq_{C_1} \lambda F_1(y,y_1',x) + (1-\lambda)F_1(y,y_2',x)$. This implies $\langle \xi_1, F_1(y,\lambda y_1' + (1-\lambda)y_2',x)\rangle \leq \lambda\langle \xi_1, F_1(y,y_1',x)\rangle + (1-\lambda)\langle \xi_1, F_1(y,y_2',x)\rangle$, and then $\langle \xi_1, F_1(y,\lambda y_1' + (1-\lambda)y_2',x)\rangle \leq \langle \xi_1, F_1(y,v,x)\rangle$, for all $v \in T(x,y)$. Thus, $\lambda y_1' + (1-\lambda)y_2' \in H(x,y)$ and so $H(x,y)$ is convex set.

If $F_1(y,.,x)$ is C_1-quasi-convex-like, we have $F_1(y,\lambda y_1' + (1-\lambda)y_2',x) \preceq_{C_1} F_1(y,y_1',x)$, or, $F_1(y,\lambda y_1' + (1-\lambda)y_2',x) \preceq_{C_1} F_1(y,y_2',x)$.

In both the cases, we get $\langle \xi_1, F_1(y,\lambda y_1' + (1-\lambda)y_2',x)\rangle \leq \langle \xi_1, F_1(y,v,x)\rangle, \forall v \in T(x,y)$. Thus, $\lambda y_1' + (1-\lambda)y_2' \in H(x,y)$ and so $H(x,y)$ is a convex set.

Next, we claim that H is a closed multivalued mapping. Let $x_\alpha \to x, y_\alpha \to y, y_\alpha' \in H(x_\alpha,y_\alpha), y_\alpha' \to y'$. We have to show that $y' \in H(x,y)$. Indeed, since $y_\alpha' \in T(x_\alpha,y_\alpha)$ and the upper semicontinuity and the closed values of T, we conclude that $y' \in T(x,y)$. For $y_\alpha' \in H(x_\alpha,y_\alpha)$, we have $\langle \xi_1, F_1(y_\alpha,y_\alpha',x_\alpha)\rangle \leq \langle \xi_1, F_1(y_\alpha,v,x_\alpha)\rangle$, for all $v \in T(x_\alpha,y_\alpha)$.

For each $v \in T(x,y)$, by the lower semicontinuity of T, there exists $v_\alpha \in T(x_\alpha,y_\alpha)$ such that $v_\alpha \to v$. We have $\langle \xi_1, F_1(y_\alpha,y_\alpha',x_\alpha)\rangle \leq \langle \xi_1, F_1(y_\alpha,v_\alpha,x_\alpha)\rangle, \forall\ \alpha$. Since F_1 is a $(-C_1)$-continuous and C_1- continuous multivalued mapping, there exists an index α_0 such that, for all $\alpha \geq \alpha_0$, we have

$$\begin{aligned} F_1(y_\alpha,v_\alpha,x_\alpha) &\in F_1(y,v,x) + V - C_1; \\ F_1(y,y',x) &\in F_1(y_\alpha,y_\alpha',x_\alpha) + V - C_1 \end{aligned}$$

and so,

$$\begin{aligned} \langle \xi_1, F_1(y_\alpha,v_\alpha,x_\alpha)\rangle &< \langle \xi_1, F_1(y,v,x)\rangle + \frac{\epsilon}{2}; \\ \langle \xi_1, F_1(y,y',x)\rangle &< \langle \xi_1, F_1(y_\alpha,y_\alpha',x_\alpha)\rangle + \frac{\epsilon}{2}. \end{aligned}$$

Hence, $\langle \xi_1, F_1(y,y',x)\rangle < \langle \xi_1, F_1(y,v,x)\rangle + \epsilon$. Thus, $\langle \xi_1, F_1(y,y',x)\rangle \leq \langle \xi_1, F_1(y,v,x)\rangle, \forall v \in T(x,y)$.

This implies $y' \in H(x,y)$ and H is a closed multivalued mapping. Then, it follows from the compactness of the subset K that H_1 is upper semi-continuous with nonempty closed convex values. The subset $A = \{(x,y) \mid x \in S(x,y), y \in T(x,y)\}$ is closed and so is the subset $\{(x,y) \mid x \in S(x,y), y \in H(x,y)\}$. Indeed, assume that $x_\alpha \to x, y_\alpha \to y, x_\alpha \in S(x_\alpha, y_\alpha), y_\alpha \in H(x_\alpha, y_\alpha)$. This implies that $x_\alpha \in S(x_\alpha, y_\alpha), y_\alpha \in T(x_\alpha, y_\alpha)$. Since A is closed, we conclude that $x \in S(x,y)\ y \in T(x,y)$. Since H is a closed multivalued mapping and $y_\alpha \in H(x_\alpha, y_\alpha); (x_\alpha, y_\alpha) \to (x,y)$, we get $y \in H(x,y)$. Therefore, $x \in S(x,y), y \in H(x,y)$.

Lastly, we define the multivalued mapping $M : D \to 2^D$ by $M(x) = \{t \in P(x) | \langle \xi_2, F_2(y,x,x)\rangle > \langle \xi_2, F_2(y,x,t)\rangle, \text{ for some } y \in Q(x,t)\}$.

We verify that the multivalued mapping M has open lower sections and for any $x \in D, x \notin coM(x)$. Indeed, we can see that

$M(x) = \{t \in D | \langle \xi_2, F_2(y,x,x)\rangle > \langle \xi_2, F_2(y,x,t)\rangle, \text{ for some } y \in Q(x,t)\} \cap P(x)$.

For any $t \in D$, deduce that

$M^{-1}(t) = \{x \in D | \langle \xi_2, F_2(y,x,x)\rangle > \langle \xi_2, F_2(y,x,t)\rangle, \text{ for some } y \in Q(x,t)\} \cap P^{-1}(t)$.

Setting $B(t) = \{x \in D | \langle \xi_2, F_2(y,x,x)\rangle > \langle \xi_2, F_2(y,x,t)\rangle, \text{ for some } y \in Q(x,t)\}$,
we first show that $B(t)$ is open in D. One can easily verify that $D \setminus B(t) = \{x \in D | \langle \xi_2, F_2(y,x,x)\rangle \leq \langle \xi_2, F_2(y,x,t)\rangle, \text{ for all } y \in Q(x,t)\}$.

Let $x_\alpha \in D \setminus B(t)$ and $x_\alpha \to x$. We have to show that $x \in D \setminus B(t)$. Indeed, since $x_\alpha \in D \setminus B(t)$ we conclude that $\langle \xi_2, F_2(y, x_\alpha, x_\alpha)\rangle \leq \langle \xi_2, F_2(y, x_\alpha, t)\rangle, \text{ for all } y \in Q(x_\alpha, t)$.

Take an arbitrary $y \in Q(x,t)$. Since $Q(.,t) : D \to 2^K$ is lower semicontinuous, there exists $y_\alpha \in Q(x_\alpha, t)$ with $y_\alpha \to y$. The C_2-continuity of N_2 and the $(-C_2)-$continuity of $F_2(.,.,t)$ follow that for any neighborhood V of the origin in Y_2, there exists α_1 such that, for all $\alpha \geq \alpha_1$,

$$\begin{aligned} F_2(y,x,x) &\in F_2(y_\alpha, x_\alpha, x_\alpha) + V - C_2, \\ F_2(y_\alpha, x_\alpha, t) &\in F_2(y,x,t) + V - C_2. \end{aligned}$$

This gives

$\langle \xi_2, F_2(y,x,x)\rangle \leq \langle \xi_2, F_2(y,x,t)\rangle, \forall\, y \in Q(x,t)$, and hence $x \in D \setminus B(t)$. Thus, $D \setminus B(t)$ is a closed subset in D and then $B(t)$ is an open subset in D. Therefore, $M^{-1}(t) = B(t) \cap P^{-1}(t)$ is open for any $t \in D$. Consequently, M has open lower sections.

Further, assume that there is $x \in D$ such that $x \in coM(x)$. Then there exists $t_1, t_2, ..., t_n \in M(x)$ such that $x = \sum_{i=1}^{n} \alpha_i t_i, \alpha_i \geq 0, \sum_{i=1}^{n} \alpha_i = 1$. This yields $\langle \xi_2, F_2(y_i, x, x)\rangle > \langle \xi_2, F_2(y_i, x, t_i)\rangle, \text{ for some } y_i \in Q(x, t_i)$. (3.5)

On the other hand, if $F_2(y,.,.)$ is diagonally C_2-convex in the second variable, we have $F_2(y,x,x) \in \sum_{j=1}^{n} \alpha_j F_2(y,x,t_j) - C_2, \forall\, y \in K$.

This yields

$\langle \xi_2, F_2(y,x,x)\rangle \leq \max\limits_{z \preceq_{C_2} \sum_{j=1}^{n} \alpha_j F_2(y,x,t_j)} \langle \xi_2, z\rangle \leq \max\limits_{j=1,\dots,n} \langle \xi_2, F_2(y,x,t_j)\rangle, \text{ for all } y \in K$.

This contradicts (3.5).

If $F_2(y,.,.)$ is diagonally C_2-quasi-convex-like in the second variable, then there is an index $j \in \{1,...,n\}$ it holds $F_2(y,x,x) \preceq_{C_2} F_2(y,x,t_j), \text{ for all } y \in K$. This implies

$\langle \xi_2, F_2(y,x,x)\rangle \leq \langle \xi_2, F_2(y,x,t_j)\rangle, \forall\, y \in K$.

This also contradicts (3.5).

Therefore, for both the cases, we conclude that $x \notin coM(x)$ for any $x \in D$.

Thus, S, T, H and M satisfy all assumptions of Lemma 2.4 in Section 2. Applying this theorem, we conclude that there exists $(\bar{x}, \bar{y}) \in D \times K$ such that $\bar{x} \in S(\bar{x}, \bar{y}), \bar{y} \in H(\bar{x}, \bar{y})$ and $S(\bar{x}, \bar{y}) \cap M(\bar{x}) = \emptyset$. This implies $\bar{x} \in S(\bar{x}, \bar{y}), \bar{y} \in T(\bar{x}, \bar{y}),\ \langle \xi_1, F_1(\bar{y}, \bar{y}, \bar{x})\rangle \leq \langle \xi_1, F_1(\bar{y}, v, \bar{x})\rangle, \text{ for all } v \in T(\bar{x}, \bar{y})$. (3.6)

And, for any $t \in P(\bar{x})$ it shows $t \in S(\bar{x}, \bar{y})$ and $t \notin M(\bar{x})$. Therefore, we get $\langle \xi_2, F_2(y, \bar{x}, \bar{x})\rangle \leq \langle \xi_2, F_2(y, \bar{x}, t)\rangle, \text{ for all } y \in Q(\bar{x}, t)$.

We now show that $F_1(\bar{y}, v, \bar{x}) \succeq_{C_1} F_1(\bar{y}, \bar{y}, \bar{x}), \text{ for all } v \in T(\bar{x}, \bar{y})$.

Assume that there exists $v^* \in T(\bar{x}, \bar{y})$ such that $F_1(\bar{y}, v^*, \bar{x}) \prec F_1(\bar{y}, \bar{y}, \bar{x})$. This implies $\langle \xi_1, F_1(\bar{y}, v^*, \bar{x})\rangle < \langle \xi_1, F_1(\bar{y}, \bar{y}, \bar{x})\rangle$.

This contradicts (3. 6).

Thus, $\bar{x} \in S(\bar{x}, \bar{y}), \bar{y} \in T(\bar{x}, \bar{y})$,

$F_1(\bar{y}, v, \bar{x}) \succeq_{C_1} F_1(\bar{y}, \bar{y}, \bar{x}),$ for all $v \in T(\bar{x}, \bar{y})$.
Analogically, we obtain
$F_2(y, \bar{x}, t) \succeq_{C_2} F_2(y, \bar{x}, \bar{x})$, for all $t \in P(\bar{x}), y \in Q(\bar{x}, t)$,
and the proof is completed. □

Remark 3.2 *We assume that all the hypotheses of Theorem 3.1 are satisfied except for (i) and (iii) (respectively) replaced by*

(i') S is a lower semi-continuous multivalued mapping with nonempty convex values;

(iii') P is lower semi-continuous and $P(x) \subseteq S(x, y)$ for all $x \in S(x, y), y \in T(x, y)$ and the subset $A = \{(x, y) \in D \times K | (x, y) \in S(x, y) \times T(x, y)\}$ is closed.

Then the inequalities of that theorem is also true.

Next, given multivalued mappings S, T with nonempty values, F_1, F_2 as in Introduction, we are interested in the system of quasi-optimization problems of Types 1.

4 Applications

4.1 System of two Pareto quasi-optimization problems of type 1.

Find $(\bar{x}, \bar{y}) \in D \times K$ such that

$\bar{x} \in S(\bar{x}, \bar{y}); \bar{y} \in T(\bar{x}, \bar{y});$
$F_1(\bar{y}, \bar{y}, \bar{x}) \in PMin(F_1(\bar{y}, T(\bar{x}, \bar{y}), \bar{x})|C_1);$
$F_2(\bar{y}, \bar{x}, \bar{x}) \in PMin(F_2(\bar{y}, \bar{x}, S(\bar{x}, \bar{y})|C_2).$
We have the following results:

Theorem 4.1 *Assume that the following conditions hold:*

(i) D, K are nonempty convex compact subsets;

(ii) S and T are continuous multivalued mappings with nonempty closed convex values;

(iii) The mapping F_i is a $(-C_i)-$ continuous and C_i- continuous mapping.

(iv) For any fixed $(x, y) \in D \times K$, the mapping $F_1(y, ., x) : K \to 2^{Y_1}$ is C_1- convex (or, C_1-quasi-convex-like) and mapping $F_2(y, ., .) : D \times D \to 2^{Y_2}$ is lower C_2-convex (or, C_2-quasi-convex-like).

Then there exists $(\bar{x}, \bar{y}) \in D \times K$ such that $\bar{x} \in S(\bar{x}, \bar{y}); \bar{y} \in T(\bar{x}, \bar{y});$ such that there are no $v \in T(\bar{x}, \bar{y}), v \neq \bar{y}, t \in S(\bar{x}, \bar{y}), t \neq \bar{x}$ with
$F_1(\bar{y}, \bar{y}, \bar{x}) \succeq_{C_1} F_1(\bar{y}, v, \bar{x});$
$F_2(\bar{y}, \bar{x}, \bar{x}) \succeq_{C_2} F_2(\bar{y}, \bar{x}, t).$

Proof. Let $\xi_i \in C_i'^{+}, i = 1, 2$ be fixed. Let $\epsilon > 0$ be arbitrary. Since ξ_i is continuous, there exists a neighborhood V of the origin in Y such that $\xi_i(V) \subseteq (-\frac{\epsilon}{2}, \frac{\epsilon}{2})$. We define the multivalued mapping $H_i : D \times K \to 2^K, i = 1, 2$ by
$H_1(x, y) = \{y' \in T(x, y) : \langle \xi_1, F_1(y, y', x) \rangle \leq \langle \xi_1, F_1(y, v, x) \rangle,$ for all $v \in T(x, y)\}$.
$H_2(x, y) = \{x' \in S(x, y) : \langle \xi_2, F_2(y, x, x') \rangle \leq \langle \xi_2, F_1(y, x, t) \rangle,$ for all $t \in S(x, y)\}$.
By the same arguments as in the proof of Theorem 3.1, we conclude that $H_i, i = 1, 2$ are upper semi-continuous multivalued mappings with nonempty convex and compact values.
Further, we define multivalued mapping $G : D \times K \to 2^{D \times K}$ by
$G(x, y) = H_2(x, y) \times H_1(x, y), (x, y) \in D \times K$.
Then, G is also a upper semi-continuous multivalued mapping with nonempty convex and compact values. According to Ky Fan fixed point Theorem there exists $(\bar{x}, \bar{y}) \in D \times K$ such that $(\bar{x}, \bar{y}) \in G(\bar{x}, \bar{y})$. This yields $\bar{x} \in H_2(\bar{x}, \bar{y})$ and $\bar{y} \in H_1(\bar{x}, \bar{y})$ and then

$$\bar{x} \in S(\bar{x}, \bar{y}); \bar{y} \in T(\bar{x}, \bar{y});$$
$$F_1(\bar{y}, v, \bar{x}) \succeq_{C_1} F_1(\bar{y}, \bar{y}, \bar{x}), \forall v \in T(\bar{x}, \bar{y});$$
$$F_2(\bar{y}, \bar{x}, t) \succeq_{C_2} F_2(\bar{y}, \bar{x}, \bar{x}) - (C_2 \setminus \{0\});$$
$$\text{for all } t \in S(\bar{x}, \bar{y}).$$

Therefrom, the proof is completed. □

To conclude this paper, we assume that $F_1(y, y, x) \succeq_{C_1} 0$ and $F_2(y, x, x) \succeq_{C_2} 0$ for any $(x, y) \in D \times K$, then we obtain the above theorem for mixed Pareto quasi-equilibrium problems.

4.2 Mixed Pareto quasi-equilibrium problems

Theorem 4.2 *We assume that the following conditions hold:*

(i) D, K *are nonempty convex compact subsets;*

(ii) S *is a multivalued with nonempty convex values and has open lower sections and* T *is a continuous multivalued mapping with nonempty closed convex values and the subset* $A = \{(x,y) \in D \times K | (x,y) \in S(x,y) \times T(x,y)\}$ *is closed;*

(iii) P *has open lower sections and* $P(x) \subseteq S(x,y)$ *for* $(x,y) \in A$*. For any fixed* $t \in D$*, the multivalued mapping* $Q(.,t) : D \to 2^K$ *is lower semi-continuous with compact values;*

(iv) The mapping F_1 *is a* $(-C_1)-$ *continuous and* C_1- *continuous mapping. The mapping* F_2 *is a* $(-C_2)$*- continuous mapping and for any fixed* $y \in Y$*, the mapping* $N_2 : K \times D \to Y_2$ *defined by* $N_2(y,x) = F_2(y,x,x)$ *is* C_2-*continuous ;*

(v) For any fixed $(x,y) \in D \times K$*, the mapping* $F_1(y,.,x) : K \to Y_1$ *is* C_1- *convex (or,* C_1-*quasi-convex-like) and any* $y \in K$ *the mapping* $F_2(y,.,.) : D \times D \to Y_2$ *is diagonally* C_2*-convex in the second variable (or, diagonally* C*-quasi-convex-like in the second variable);*

(vi) $F_1(y,y,x) \succeq_{C_1} 0$ *and* $F_2(y,x,x) \succeq_{C_2} 0$ *for any* $(x,y) \in D \times K$*.*

Then there exists $(\bar{x},\bar{y}) \in D \times K$ *such that* $\bar{x} \in S(\bar{x},\bar{y}); \bar{y} \in T(\bar{x},\bar{y})$ *and there are no* $v \in T(\bar{x},\bar{y}), v \neq \bar{y}, t \in P(\bar{x}), y \in Q(\bar{x},t), t \neq \bar{x}$ *such that*
$F_1(\bar{y},v,\bar{x}) \succeq_{C_1} 0$ *and* $F_2(y,\bar{x},t) \succeq_{C_2} 0.$

References

[1] Tấn, N. X. và Minh, N. B., *Một số vấn đề trong lý thuyết tối ưu véctơ đa trị*, NXBGD, 2006.

[2] Balaj, M and Luc, D.T., *On mixed variational relation problems*, Computers and Mathematic with Applications 60(9), 2010, 2712-2722.

[3] Duong, T. T. T. and Tan, N. X., *On the existence of solutions to generalized quasi-equilibrium problems of type I and Related Problems.* Adv Nonlinear Var Inequal, 13, No. 1, 2010, 29-47.

[4] Ferro, F., *A minimax theorem for vector-valued functions.* J. Optim. Theory and Appl, 60, 1989, 19-31.

[5] Luc, D. T., *An abstract problem in variational analysis* , J. Optim. Theory Appl, 138, 2008, 65-76.

[6] Minh, N. B. and Tan, N. X., *On the existence of solutions of quasi-variational inclusion problems of Stampacchia type.* Adv. Nonlinear Var. Inequal, 8, 2005, 1–16.

[7] Sach, P. H. and Tuan, L. A., *Existence Results for Set-valued Vector Quasiequilibrium Problems.* J. Optim Theory and Appl, 133, 2007, 229-240.

TÓM TẮT

SỰ TỒN TẠI NGHIỆM CỦA BÀI TOÁN TỰA TỐI ƯU PARETO HỖN HỢP PHỤ THUỘC THAM SỐ

Bài báo đưa ra bài toán tựa tối ưu Pareto hỗn hợp và chỉ ra điều kiện đủ để bài toán đó có nghiệm. Trong trường hợp đặc biệt, bài báo chỉ ra sự tồn tại nghiệm của bài toán tựa cân bằng Pareto hỗn hợp và hệ tựa tối ưu Pareto.

Từ khóa. *Bài toán tựa tối ưu Pareto hỗn hợp, ánh xạ C-lồi, ánh xạ C-giống như tựa lồi, ánh xạ C-liên tục, ánh xạ C-lồi (giống như tựa lồi) theo đường chéo.*

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